

## E3-brane instantons and baryonic operators for D3-branes on toric singularities

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## E3-brane instantons and baryonic operators for D3-branes on toric singularities

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**ABSTRACT:** We consider the couplings induced on the world-volume field theory of D3-branes at local toric Calabi-Yau singularities by euclidean D3-brane (E3-brane) instantons wrapped on (non-compact) holomorphic 4-cycles. These instantons produce insertions of BPS baryonic or mesonic operators of the four-dimensional  $\mathcal{N} = 1$  quiver gauge theory. We argue that these systems underlie, via the near-horizon limit, the familiar AdS/CFT map between BPS operators and D3-branes wrapped on supersymmetric 3-cycles on the 5d horizon. The relation implies that there must exist E3-brane instantons with appropriate fermion mode spectrum and couplings, such that their non-perturbative effects on the D3-branes induce operators forming a generating set for all BPS operators of the quiver CFT. We provide a constructive argument for this correspondence, thus supporting the picture.

**KEYWORDS:** Solitons Monopoles and Instantons, AdS-CFT Correspondence, Brane Dynamics in Gauge Theories, 1/N Expansion

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**1 Introduction**

The generalization of the AdS/CFT correspondence to dual pairs related to D3-branes at singularities [1–3] has provided many new insights into the duality in situations of reduced supersymmetry (for instance, [4–11]) or broken conformal invariance (for instance [12–17]). Progress has been particularly significant for toric Calabi-Yau threefold singularities, for which there exist powerful tools to study both the field theory and the CY geometry, like dimer diagrams (aka brane tilings) [18–21], see [22, 23] for reviews. One of the most active topics in this direction is the identification of gravity duals of the BPS operators of the CFT and the derivation of BPS operator counting techniques [24–37].

BPS operators with low conformal dimension are usually regarded as dual to supergravity modes [38]. However the systematic discussion of general BPS operators, including those with a number of fields comparable with the number of D3-branes  $N$ , is most conveniently carried out by considering all BPS operators to be dual to systems of supersymmetric D3-branes on the 5d horizon [39]. These are generalizations of the familiar giant gravitons [40], and of basic determinant operators [41–44]. Since most such operators carry non-trivial charges under the baryonic symmetries of the quiver theory we refer to them as baryonic operators. Note that in this language mesonic operators are a subset of baryonic operators having vanishing baryonic charge. The correspondence between BPS baryonic operators and supersymmetric wrapped D3-branes has been mostly based on a precise matching of conformal dimensions and quantum numbers between the two kinds of objects. Namely, without a more dynamical explanation of the fact that baryonic operators correspond to wrapped D3-brane states.

In this paper we provide a dynamical understanding of the realization of the gravity dual of baryonic operators in terms of wrapped D3-branes. Moreover the explanation involves consideration of euclidean D-brane instantons, concretely E3-branes wrapped on holomorphic 4-cycles of the CY in the presence of the gauge D3-branes. In crude terms, the E3-brane instantons leads to insertions of baryonic operators in the gauge D3-branes, at the level of the system of D3-branes on the CY geometry. The near horizon version of the map is that the BPS baryonic operators is related to the boundary behaviour of the E3-brane, which corresponds to a D3-brane wrapped on a supersymmetric 3-cycle. The argument is tightly related to the very suggestive fact [45], already exploited in the literature, that supersymmetric D3-branes on the horizon can be characterized in terms of holomorphic 4-cycles on the CY singularity.

The holomorphic 4-cycles on which we wrap the E3 instantons are non-compact, and thus one would say that the instanton action vanishes. We will assume the existence of some effective cutoff for the volume of the cycle, generically given by the compactification of the local geometry we are studying, and we will just be interested on the prefactor that gives the field theory operator induced by the instanton, without entering into details of how the setup could be embedded globally. Which field theory operator is inserted can be determined by a purely local analysis near the D3 branes.

Note that in the above argument, the E3-brane instantons are considered dynamical, in the sense that its non-perturbative effect is considered as included in the discussion. This is in contrast with the recent use of E3-branes on 4-cycles as probes of vevs for baryonic operators [46, 47]. However there is no contradiction, but rather a nice agreement, between the two interpretations of E3-branes on 4-cycles. It is the analog of the familiar statement [48] that a given AdS field encodes the information about both the insertion of operators deforming the CFT, and about the vev of the operator in a given CFT vacuum dual to a given gravity background. The latter is determined by the normalizable mode of the AdS field, namely, the component decaying at the boundary, and can be detected by considering a probe fluctuation of the field and evaluating its action. Similarly, in order to measure the vev for a baryonic operator in a given gravity background, one can introduce a probe with the appropriate asymptotics, namely given by a D3-branes on a 3-cycle. The

corresponding probe is an E3-brane wrapped a holomorphic 4-cycle on the CY geometry, and the exponential of its action measures the vev. This is similar to the computation of a Wilson loop by a worldsheet with appropriate asymptotics.

The relation between E3-brane instanton effects on D3-branes at CY singularities and BPS operators in AdS/CFT has a direct implication: the set of BPS operators in the quiver CFT which can be generated from non-perturbative effects of BPS E3-brane instantons on the CY must form a generating set of all CFT BPS operators. Also the boundary of a given E3-brane instanton defines the baryonic D3-brane providing the gravity dual of the corresponding BPS operator arising from the non-perturbative effect. Equivalently, the E3-brane on the holomorphic 4-cycle corresponding to the baryonic D3 (i.e. constructed as a cone over the wrapped horizon 3-cycle) must have a specific structure of fermion modes charged under the D3-brane theory, and with appropriate E3-brane world-volume couplings. In this paper we provide a systematic (and constructive) derivation of this result, for systems related to D3-branes at toric singularities. This result provides a strong support for our picture of E3-brane instanton effects as a first-principle derivation of the AdS/CFT relation between BPS operators and wrapped D3-branes in AdS/CFT, and of the use of E3-branes as probes of baryonic vevs.

Let us finish this introduction by remarking that the discussion in this paper is one instance of a very general and deep relation between instantons in 5 dimensions and baryons, and can be traced back to early studies of baryons as solitons in the Skyrme model [49]. More recently, this connection has also been realized in the context of Sakai-Sugimoto models for holographic QCD [50]. The results of this paper generalize this correspondence to the rich class of theories arising from D3 branes at toric singularities.

This paper is organized as follows. In section 2 we describe a basic example of the role of E3-brane instantons in systems of D3-branes in local CY geometries, and its implication for the near horizon AdS/CFT relation between baryonic operators and wrapped D3-brane states. In section 3 we review the construction of general BPS operators and their dual wrapped D3-brane states in AdS/CFT, for systems of branes at singularities. We discuss the conifold example explicitly, and provide the generalization to arbitrary toric singularities. In section 4 we describe the generation of general BPS 4d field theory operators by E3-brane instantons, for systems with a single D3-brane. The arguments involve diverse geometric/field theory operations, such as orbifolding, partial resolution/Higgsing, as well as a very geometric discussion in terms of the mirror configuration of E2-brane instantons on systems of intersecting D6-branes. Our analysis shows a one-to-one map between field theory BPS operators and 4-cycles on which E3-brane instantons wrap, which exactly reproduces the AdS/CFT relation. In section 5 we describe the generalization to arbitrary number of D3-branes, where the map between operators and 4-cycles is more involved in a sense that we make precise. Finally in section 6 we present our final comments.

## 2 E3-brane instantons and baryonic D3-branes

Let us consider a configuration of type IIB D3-branes, spanning 4d Minkowski space  $\mathbf{R}^{1,3}$  and sitting at the singular point of Calabi-Yau threefold geometry. The gauge theory

on the D3-brane world-volume is determined by the local structure of the singularity at which the D3-branes sit. We consider the local singularity to be given by a real cone  $C(H)$  over a Sasaki-Einstein 5d manifold  $H$ . The low energy dynamics of these branes is a four dimensional  $\mathcal{N} = 1$  supersymmetric gauge theory with gauge group  $\prod_i \text{SU}(N_i)$ , and a set of chiral multiplets in bifundamental representations, see e.g. [8–10, 20] for details on the construction of the field theory from geometric data of the singularity. We adopt the viewpoint that all  $U(1)$  factors (except a decoupled one, which we ignore) are massive due to  $B \wedge F$  couplings with RR 2-forms, and are therefore absent from the low-energy dynamics. The  $N_i$  are positive integers, subject to the condition of anomaly cancellation or cancellation of localized RR tadpoles. In this paper we focus on toric singularities, and in the conformal case  $N_i \equiv N$  which automatically satisfies these constraints.

This type of local systems of D3-branes at CY singularities plays an important role in two contexts, as local models of type IIB compactifications to four dimensions, and in the gauge/gravity correspondence. The latter can be regarded as the near-horizon limit of the former, leading to derivations of certain results in AdS/CFT. For instance, the fact that a given AdS field  $\phi$  is dual to certain operator  $\mathcal{O}$  in the holographic field theory can be obtained from the fact that in the original system of D3-branes on CY, there is a D3-brane world-volume coupling  $\phi \mathcal{O}$ .

In this section we argue that one can draw a similar relation between baryonic BPS operators in the holographic field theory and AdS particles from D3-branes on 3-cycles on the horizon, by considering E3-brane instanton effects on the initial system of D3-branes in a singular CY geometry.

Let us consider a configuration of D3-branes at a local CY singularity. It is a natural question to consider the structure of field theory operators that can be induced by non-perturbative effects in this setup. There are instanton effects, coming from wrapped euclidean D-branes [51–54] (denoted E-branes henceforth) which can induce interesting field theory operators [55–57]. In our setup, BPS instantons preserving half of the 4d  $\mathcal{N} = 1$  supersymmetry arise from E3-branes wrapped on holomorphic 4-cycles in the internal space.<sup>1</sup> In the non-compact setup, one should distinguish between E3-branes wrapped on compact or non-compact 4-cycles. The E3 branes wrapped on compact cycles are classified by the nodes of the quiver, and correspond to gauge theory instantons when the node is filled by two or more 4d gauge branes. Even if there is just one or no 4d gauge branes filling the corresponding node, one can use field theory techniques to understand the properties of the instanton, see e.g. [58–60]. We focus instead on E3-branes wrapped on non-compact 4-cycles, passing through (or near) the singularity, so that they survive in the near horizon limit to be taken later on. Note that our setup is a generalization of that recently considered in [61], with emphasis on a different motivation.

In the non-compact setup these instantons have vanishing strength, but such instanton effects become physical when the local model is embedded in a full-fledged compactification. Some of the properties of the instanton depend on the global structure of the 4-cycle in the compactification. For instance, the kind of 4d superspace interaction they induce is

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<sup>1</sup>There are also  $E(-1)$  instantons, that we will not consider.

determined by the number of unlifted fermion zero modes of the instanton. For simplicity, we will assume that the instantons have only two uncharged fermion zero modes in an appropriate compactification (the goldstinos of the two 4d  $\mathcal{N} = 1$  supersymmetries they break) and therefore generate a non-perturbative superpotential (with the measure  $d^2\theta$  saturating the two fermion zero modes). Note that this imposes some specific constraints on the D3-brane, e.g. to be invariant under the orientifold action on the compactification, with an  $O(1)$  Chan-Paton symmetry. This will not be very important for our analysis, and in fact the presence of additional fermion zero modes will simply lead to the insertion of additional operators  $\overline{D}\Phi$  in the resulting multi-fermion F-term, as studied in [62, 63], see also [64] for a recent discussion.

Rather, our interest lies in the D3-brane field theory couplings induced by the non-perturbative instanton effect. The basic structure of this coupling essentially depends only on the local properties of the configuration, since it arises from the integration of the charged fermion modes in the D3-E3 open string sector. These zero modes appear in the instanton world-volume action via couplings to (combinations of the) bi-fundamental fields of the 4d field theory, and integration over them leads to the insertion of a BPS operator of the world-volume D3-brane field theory. The detailed mapping between E3-branes and BPS operators will be discussed in coming sections, but it is useful to present now the basic idea. Consider an E3-brane wrapped on a 4-cycle passing through the system of D3-branes. The E3-D3 open string sector leads to charged fermion zero modes  $\alpha_{\tilde{i}}, \beta_j$ , where  $\tilde{i}, j$  are gauge indices. These fields transform as  $\overline{\square}_a, \square_b$ , respectively, of the  $SU(N)_a \times SU(N)_b$  factor of the D3-brane gauge theory. They couple to a 4d chiral multiplet  $\Phi^{ab}$  in the  $(\square_a, \overline{\square}_b)$  in the instanton action, as

$$\Delta S_{E3} = \alpha_{\tilde{i}} \Phi_{\tilde{i}j}^{ab} \beta_j \tag{2.1}$$

The detailed structure of zero modes and the form of the coupling can be deduced, as we will argue in detail in section 4, from the cycle wrapped by the instanton and its Chan-Paton factors. Integrating over the fermion zero modes (and assuming no extra fermion zero modes beyond the two goldstinos), the instanton leads to a 4d superpotential

$$\delta_{\text{inst}} W \simeq e^{-T} \det \Phi^{ab} \tag{2.2}$$

where  $T$  denotes the modulus associated to the 4-cycle in an eventual global embedding of the local configuration, and where the determinant contracts the color indices, as

$$\det \Phi^{ab} = \frac{1}{N!} \epsilon_{i_1 \dots i_N} \epsilon_{\tilde{j}_1 \dots \tilde{j}_N} (\Phi^{ab})_{i_1 \tilde{j}_1} \dots (\Phi^{ab})_{i_N \tilde{j}_N} \tag{2.3}$$

Hence, the above instanton computation leads to a connection between 4-cycles in the singular geometry and BPS (di)baryonic operators in the 4d field theory. This is an example of the general correspondence to be studied in sections 4 and 5.

Let us now connect the above discussion to the usual AdS/CFT discussion for baryonic operators. Consider the near horizon limit of the above system of D3-branes placed on the singularity of  $C(H)$ . As discussed in [2, 3] it corresponds to type IIB on  $\text{AdS}_5 \times H$ , with  $N$  units of RR 5-form flux on  $H$ . The AdS/CFT implies that this background is exactly

equivalent to the CFT arising from the world-volume D3-brane field theory considered above. The precise dictionary relates operators  $\mathcal{O}$  of the CFT to AdS fields  $\phi$ , in a way that can in many cases be derived from the existence of a coupling  $\phi\mathcal{O}$  in the original D3-brane world-volume field theory. In this sense, it is natural to expect that the dual of the BPS baryonic operators is related to E3-branes on CY 4-cycles. In order to make this manifest, recall that the source for the CFT operator  $\mathcal{O}$  is given by the asymptotic boundary configuration of the AdS object which produces its coupling. Thus we may expect that the source for the BPS baryonic operators is given by the asymptotic boundary configuration of the E3-brane on the CY holomorphic 4-cycle. The near horizon structure of a holomorphic 4-cycle is a conical 4-cycle  $C(S)$  whose base is a 3-cycle. The state providing the dual to the baryonic BPS operator  $\det\Phi^{ab}$  is thus an AdS particle given by D3-brane wrapped on the 3-cycle  $S$  on the horizon.<sup>2</sup> This therefore reproduces (and in a sense, explains) the familiar relation between BPS operators and wrapped D3-branes, and the relevant role played by holomorphic 4-cycles in their construction [39], see [30, 33, 34].

The above is just an example of a more general correspondence (which includes BPS mesonic operators as well), which we establish in detail in this paper. For each BPS operator  $\mathcal{O}$  (in a suitable generating set of all BPS operators) in the CFT there exists an E3-brane instanton wrapped on a holomorphic 4-cycle on the local CY geometry, such that the non-perturbative instanton amplitude induces an insertion of the operator  $\mathcal{O}$  in the D3-brane world-volume theory. This requires a specific structure of fermion zero modes and couplings to the CFT fields, which we clarify in sections 4 and 5.

As mentioned in the introduction, the effect of the E3-brane instanton on the 4-cycle  $C(S)$  in the singular CY  $C(H)$  leads to an underlying explanation for two tools which are widely used in AdS/CFT:

1. The interpretation of a D3-brane wrapping the 3-cycle  $S$  on the horizon  $H$  as the gravity dual of the CFT operator  $\mathcal{O}$ , and thus the general map between BPS operators and supersymmetric wrapped D3-branes.
2. The use of E3-brane probes to measure baryonic condensates, since these probes provide configurations which asymptote to the baryonic D3-brane states in the previous point.

### 3 Wrapped branes in AdS/CFT and BPS operators

In this section we review the construction of BPS operators in quiver gauge theories for D3-branes at toric singularities, and the description of the dual states in AdS/CFT in terms of supersymmetric D3-branes wrapped on 3-cycles, following [39]. The latter are easily characterized in terms of non-compact 4-cycles of the singular geometry. We will use the conifold as illustrative example, but simultaneously discuss the generalization to arbitrary toric Calabi Yau singularities.

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<sup>2</sup>By an argument similar to [39], we can argue that the asymptotic piece of the E3-brane has a Lorentzian continuation to the wrapped D3-brane particle.



### 3.1 Symplectic quotient construction and baryonic charges

The conifold variety  $\mathbf{X}$  is usually described as the quadric  $xy - wz = 0$  in  $\mathbb{C}^4$ , but it can be equivalently described as a symplectic quotient in the following way. Let us introduce the four complex variables  $x_r$  with  $r = 1, \dots, 4$ . If we give them the charges  $(1, -1, 1, -1)$  under a  $\mathbb{C}^*$  action we can write the conifold as the holomorphic quotient

$$\mathbf{X} = \mathbb{C}^4 / (1, -1, 1, -1) \tag{3.1}$$

In terms of a symplectic quotient, this corresponds to imposing the real D-term constraint

$$|x_1|^2 + |x_3|^2 - |x_2|^2 - |x_4|^2 = 0 \tag{3.2}$$

and quotienting by the  $U(1)$  action in the above  $\mathbb{C}^*$ . To recover the usual equation for the conifold we consider a basis of the  $\mathbb{C}^*$  invariant monomials  $x = x_1x_2$ ,  $y = x_3x_4$ ,  $w = x_1x_4$ ,  $z = x_2x_3$ , which satisfy the constraint  $xy - wz = 0$ .

The low energy dynamics of a stack of  $N$  D3-branes at the conifold singularity is a  $SU(N) \times SU(N)$  gauge theory with bifundamental chiral fields  $A_1, A_2$ , and  $B_1, B_2$  in the  $(\square, \bar{\square})$  and  $(\bar{\square}, \square)$  respectively. The chiral fields interact with the superpotential  $W = \text{Tr}(\epsilon_{ij}\epsilon_{pq}A_iB_pA_jB_q)$ . The theory has a baryonic symmetry under which the fields  $A_i, B_i$  have charge  $+1, -1$ , respectively. This baryonic symmetry can be regarded as a global symmetry arising from a gauge  $U(1)$  symmetry in the  $U(N) \times U(N)$  theory, which has acquired a Stueckelberg mass due to a  $BF$  coupling.

The moduli space of the  $SU(N)^2$  theory contains the singular conifold (and all its possible resolutions) in the following way [48]. Let us restrict ourselves to the  $N = 1$  case for simplicity. In this case the gauge group becomes trivial, and the superpotential vanishes too. The moduli space of such a (free) theory of 4 complex fields  $A_i, B_i$  is simply  $\mathbb{C}^4$ . The Kahler quotient described above represents the way in which the singular and resolved conifolds foliate  $\mathbb{C}^4$ . Imposing the moment map

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = \xi \tag{3.3}$$

selects a particular size for the  $\mathbf{S}^2$  in the base of the conifold, given by  $\xi$ . The overall phase of the vevs for the different fields under the baryonic  $U(1)$  encodes the integral of  $C_4$  RR-form over the  $\mathbf{S}^2$ .

Notice that there exist a (one-to-one in this case) correspondence between the homogeneous coordinate  $x_r$  in the geometry and the elementary fields in the gauge theory  $A_i, B_j$ . In particular the  $\mathbb{C}^*$  action of the symplectic quotient construction is just the complexification of the baryonic symmetry in gauge theory. This is just a reflection of the familiar statement that the mesonic moduli space of a D3-brane is the transverse geometry, see [36, 37] for a recent discussion of the mesonic and baryonic moduli spaces of D3-branes at singularities.

The above structure generalizes to arbitrary toric singularities. This follows from their definition as symplectic/holomorphic quotients of  $\mathbb{C}^d$  by an abelian group  $K \sim (\mathbb{C}^*)^{d-3} \times \Delta$ , where  $\Delta$  is a discrete group. Indeed, like in the conifold case, there is a relation between

homogeneous coordinates in the symplectic quotient construction and chiral multiplets of the D3-brane gauge theory. The relation however is in general not one-to-one, and to each homogeneous coordinate in the geometry is associated more than one chiral superfield [11]. Also the D3-brane field theories have a set of baryonic symmetries, which can be regarded as the U(1) factors in the  $U(N) \times \dots \times U(N)$  theory, eventually massive by the Stueckelberg mechanism. In analogy with the conifold case, these baryonic symmetries can be related to the U(1) symmetries in the symplectic quotient construction [36, 37].

### 3.2 The general set of BPS operators

According to the AdS/CFT correspondence the low energy gauge theory of  $N$  D3-branes on  $\mathbb{R}^{1,3} \times C(H)$  is dual to string theory on the background  $AdS_5 \times H$ , for a general CY conical singularity  $C(H)$  with base a 5d Sasaki-Einstein compact manifold  $H$  [2, 3]. In particular, the AdS/CFT correspondence predicts a one-to-one map between the BPS gauge invariant operators on the field theory side and the BPS states on the gravity side.

Let us review this correspondence for the case of the conifold  $\mathbf{X} = C(T^{1,1})$ , whose gauge theory is dual to string theory on  $AdS_5 \times T^{1,1}$ . For our purposes it is useful to start by considering the simplest baryonic operators  $\det A_i, \det B_j$ .

$$\begin{aligned} \epsilon_{p_1, \dots, p_N} \epsilon^{k_1, \dots, k_N} (A_{i_1})_{k_1}^{p_1} \dots (A_{i_N})_{k_N}^{p_N} &= (\det A)_{(i_1, \dots, i_N)} \\ \epsilon_{p_1, \dots, p_N} \epsilon^{k_1, \dots, k_N} (B_{i_1})_{k_1}^{p_1} \dots (B_{i_N})_{k_N}^{p_N} &= (\det B)_{(i_1, \dots, i_N)} \end{aligned} \tag{3.4}$$

As has been studied in [42], the AdS states corresponding to these BPS operators are static D3-branes wrapping the  $S^3$  contained in the horizon manifold  $T^{1,1}$  (with a specific orientation). The specific 3-spheres are easily described using the homogeneous coordinates. Given a supersymmetric 3-cycle  $C_3$  on the horizon manifold, the real cone  $C(C_3)$  over it defines a holomorphic non-compact 4-cycle on the Calabi-Yau singular geometry, which can be described as the zero locus of the homogeneous coordinates. The baryonic operators  $\det A_i, \det B_j$  correspond to the 4-cycles  $x_r = 0$ .

This basic idea can be exploited to reproduce the full spectrum of BPS operators of the conifold theory, which includes many other operators. Indeed, the above are just the baryonic operators with the smallest possible dimension:  $\Delta_{\det A, \det B} = N \Delta_{A, B}$ . The full set of BPS operators with the same baryonic charges as e.g.  $\det A$  can be constructed as follows. Following [39, 43] (with a different notation) we define the operators

$$\mathcal{A}_{\mathcal{P}} = A_{i_1} B_{j_1} \dots A_{i_m} B_{j_m} A_{i_{m+1}} \tag{3.5}$$

Namely, we construct an operator in the  $(\square, \bar{\square})$  with the same gauge and baryonic charges as  $A$ , by concatenating a number of bifundamental fields with indices contracted, in a pattern encoded in the multi-index  $\mathcal{P}$ . In terms of the quiver we associate an operator to any path, which we also denote  $\mathcal{P}$ , obtained by concatenation of arrows corresponding to  $A$ - and  $B$ -fields.

Given a set of  $N$  (possibly different) operators of that kind, denoted  $\mathcal{A}_{\mathcal{P}_1}, \dots, \mathcal{A}_{\mathcal{P}_N}$ , we can construct the general ‘ $A$ -type’ baryonic operator as

$$\mathcal{O}_{\{\mathcal{P}\}}^A = \epsilon_{\tilde{p}_1, \dots, \tilde{p}_N} \epsilon_{k_1, \dots, k_N} (\mathcal{A}_{\mathcal{P}_1})_{p_1 \tilde{k}_1} \dots (\mathcal{A}_{\mathcal{P}_N})_{p_N \tilde{k}_N} \tag{3.6}$$

One can similarly define  $B$ -type operators. Operators (3.6) provide the generalization of the simplest baryonic operators (3.4). Note that e.g. all  $A$ -type operators carry the same baryonic charges, but are of different conformal dimension. This set of operators provides a basis of all BPS operators in the gauge theory (with mesonic operators arising from products of  $A$ - and  $B$ -type operators, so that they carry no baryonic charge, and baryonic operators of higher or lower baryonic charge coming from products of  $A$ -type and  $B$ -type operators respectively).

It is possible to generalize this discussion to general toric singularities<sup>3</sup> as follows [30]. Given one bifundamental chiral multiplet  $\Phi_{a,b}$  in the  $(\square_a, \bar{\square}_b)$ , one can form the basic di-baryonic operator generalizing (3.4) by taking its determinant  $\det(\Phi_{a,b})$ . This corresponds to the BPS operator with lowest dimension in the corresponding sector of baryonic charges. More in general, one can construct an operator with baryonic charges proportional to  $N$  under the baryonic symmetries  $U(1)_{a,b}$  (not necessarily connected by a single arrow) by considering  $N$  (possibly different) paths  $\mathcal{P}_1, \dots, \mathcal{P}_N$ , in the quiver, joining the nodes  $a, b$ .<sup>4</sup> Using the corresponding operators  $\mathcal{O}_{\mathcal{P}_1}, \dots, \mathcal{O}_{\mathcal{P}_N}$ , all of which transform in the  $(\square_a, \bar{\square}_b)$ , we can construct

$$\mathcal{O}_{\{P\}} = \epsilon_{\tilde{p}_1, \dots, \tilde{p}_N} \epsilon_{k_1, \dots, k_N} (\mathcal{O}_{\mathcal{P}_1})_{p_1 \tilde{k}_1} \cdots (\mathcal{O}_{\mathcal{P}_N})_{p_N \tilde{k}_N}. \quad (3.7)$$

Observe that, as in the conifold case, once we increase the baryonic charges we are interested in, we are forced to consider product of operators like (3.7).

### 3.3 The gravity duals and holomorphic 4-cycles

The above description is well-suited to provide a construction of the states dual to these BPS operators. Going back to the conifold example, recall that the basic baryonic operators (3.4) are mapped to static D3-branes wrapping specific three cycles of  $T^{1,1}$  in a volume-minimizing fashion. Since we would like to describe states dual to operators with the same baryonic charge but higher conformal dimension, we need to describe supersymmetric D3-branes wrapped on the same homology class, but not in a volume-minimizing fashion. The state nevertheless manages to remain BPS due to a non-trivial motion in the horizon geometry, as for the giant gravitons in [40].

These states once again have a nice correspondence with holomorphic divisors on the singular Calabi-Yau geometry. Recall that the baryonic charge of the simplest baryonic states (3.4) is related to the  $\mathbb{C}^*$  charge of the function whose zero locus defines the 4-cycle, namely  $x_i$ . Hence, the BPS operators in the same baryonic charge sector, but with higher conformal dimension, are expected to correspond to 4-cycles defined as the zero locus of a more general function of holomorphic coordinates, with the same degree of homogeneity

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<sup>3</sup>For studying gauge theories dual to D3 branes at toric singularities it is most convenient to use dimer model techniques, which also play an important role in our subsequent analysis. We include for convenience a short introduction to dimer models in appendix A.

<sup>4</sup>Since the operators are defined modulo F-terms, it is more practical to define the operator using paths joining faces in the dimer diagram. The equivalence modulo F-terms is related to the equivalence of paths under homotopy deformations. Hence different paths correspond to homotopically different paths between the faces  $a, b$ .

under the  $\mathbb{C}^*$  action. More formally, they correspond to different sections of the same non-trivial line bundle over the CY variety.

Consider for example the case of a single D3-brane  $N = 1$ , and the set of 4-cycles corresponding to BPS operators with baryonic charge  $B = 1$ . This is

$$\begin{aligned}
 f_{B=1}(x_1, x_2, x_3, x_4) = & c_1 x_1 + c_3 x_3 \\
 & + c_{11;2} x_1^2 x_2 + c_{13;2} x_1 x_3 x_2 + c_{33;2} x_3^2 x_2 \\
 & + c_{11;4} x_1^2 x_4 + c_{13;4} x_1 x_3 x_4 + c_{33;4} x_3^2 x_4 + \dots .
 \end{aligned} \tag{3.8}$$

where the coefficients, collectively denoted  $c_P$ , parametrize the complex structure of the divisor. This infinite family of holomorphic 4-cycles provides a description of all the possible supersymmetric D3-branes wrapping the  $\mathbf{S}^3$  in  $T^{1,1}$  with positive orientation. The space parametrized by the  $c_P$  is a classical configuration space for the particles arising from the D3-brane, which has to be properly quantized. Namely, the gauge theory BPS operators should correspond to appropriate wavefunctions on the space parametrized by  $c_P$ . Using geometric quantization, one can determine that the different wavefunctions are given by degree- $N$  monomials on the  $c_P$  [39]. We denote  $|c_{P_1}, \dots, c_{P_N}\rangle$  the state corresponding to the wavefunction

$$\Psi(\{c_P\}) = c_{P_1} \dots c_{P_N}. \tag{3.9}$$

This state defines a particle in  $AdS_5$ , whose dual BPS operator is obtained as follows: using the relation between monomials in  $x_r$  and bi-fundamental fields  $A, B$ , the monomial corresponding to each  $c_{P_\alpha}$  corresponds to an operator  $\mathcal{A}_{P_\alpha}$  of the form (3.5), or its B-type analog. The BPS operator dual to the state  $|c_{P_1}, \dots, c_{P_N}\rangle$  is given by the operator  $\mathcal{O}_{\{P\}}$  defined in (3.7). More general BPS operators can be generated by taking products of these.

The states in AdS side  $|c_{P_1}, \dots, c_{P_N}\rangle$  correspond to wavefunctions related to a set of  $N$  (coefficients of) such monomials in the homogeneous coordinates of  $C(H)$ . The corresponding BPS operator is a baryonic operator given by (3.7), or suitable products thereof.

This procedure extends to generic toric singularities [30]. For a general toric singularity there is also a correspondence between a monomial in the homogeneous coordinates (hence its coefficient  $c_P$  in a general expansion) and operators (denoted  $\mathcal{O}_{\mathcal{P}}$ ) given by a product of bifundamental fields describing a path  $\mathcal{P}$  in the quiver/dimer diagram of the gauge theory. The major difference with the conifold case is that in the generic case the correspondence between the homogeneous coordinate and the fields is one to many, as studied in detail in [34]. The issue here is that, if the 4-cycle wrapped by the instanton has a non-trivial homotopy group, we can construct different nontrivial flat bundles on the 4-cycle, and this information about the bundle must be specified together with the purely geometrical data in order to completely determine the map between wrapped branes and BPS operators. This makes passing from the case of the conifold to the case of the general toric singularity very nontrivial. In fact, to our knowledge, only in the orbifold case (which we discuss in detail in section 4.4) is this map well understood in terms of the explicit data of the divisor and the bundle.<sup>5</sup>

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<sup>5</sup>We thank the referee for emphasizing this point to us.

Nevertheless, in [34] a generic method to compute the multiplicities of the map from cycles to operators is proposed, and it agrees well with the field theory result for nontrivial toric singularities. This method admits a nice interpretation in the manifold mirror to the toric variety. In appendix B of [34] it is discussed how once one goes to the mirror type IIA side, the extra bundle data gets encoded into topological information of the cycles wrapping the mirror surface  $\Sigma$  (for convenience, we have included a short review of the relevant concepts in appendix A.2). We will use similar ideas in section 4.6 in order to give evidence for our results in the case of geometries with multiplicities, which are less understood from the type IIB side.

Thus, the AdS/CFT correspondence between BPS operators and wrapped D3-branes is based on associating a holomorphic 4-cycle in the CY singularity to each concatenated chain of bi-fundamentals in the field theory, in a way determined by the relation between homogeneous coordinates (plus information about the bundle) and bifundamental fields. Our proposal to provide a first principle derivation of this AdS/CFT map requires that the E3-brane instanton wrapped on the 4-cycle induces a non-perturbative insertion of precisely the dual BPS operator on the D3-brane field theory. This is explicitly shown for toric singularities in the next two sections, by a combination of techniques.

## 4 BPS operators from E3-brane instantons: the single D3-brane case

In this section we consider E3-brane instantons on non-compact holomorphic 4-cycles in general toric CY geometries, in the presence of a single D3-brane. We argue that they provide a correspondence between 4-cycles in the singular geometry and BPS operators corresponding to (part of the) 4d effective operator induced by the instanton. This correspondence is in fact nicely correlated with the one described in the previous section, lending support to our identification of E3 instantons with baryonic operators.

### 4.1 General considerations and result

Before going into details, let us summarize here the result we want to show, and the strategy that we will follow in order to show it.

In this section we will restrict the discussion to the  $N = 1$  case (here  $N$  denotes the number of branes in the singularity), which already allows us to discuss the precise form of the one-to-one map between BPS operators (and their wrapped D3-brane duals) and E3-brane instanton effects on D3-branes on the CY. We postpone the discussion of the complications arising from having  $N$  arbitrary to section 5. Although we do not provide a formal proof, we present a sufficiently general line of argument, illustrated in several explicit examples. Also, notice that the sugra approximation is expected to break down for the  $N = 1$  case, since the background will become strongly curved. Nevertheless, we expect supersymmetry to protect the BPS sector and allow the discussion in terms of holomorphic curves. Also, as we will discuss in section 5, the lessons we learn from studying this simple case in the geometric regime can be carried over easily to the regimes of larger  $N$ , where the sugra approximation is well justified.

Let us start by stating our general proposal. Since for  $N = 1$  the gauge group is trivial, the set of “single determinant” BPS operators<sup>6</sup> is described as the set of concatenated chains of bi-fundamental multiplets, modulo F-terms. Equivalently, operators carrying baryonic charges  $\pm 1$  under two baryonic symmetries  $a, b$ , are associated to paths  $\mathcal{P}_\alpha$  joining the corresponding faces in the dimer diagram, modulo homotopy transformations (see appendix A for a short review on dimer diagrams). We denote this operator by  $\mathcal{O}_{\mathcal{P}_\alpha}$ . Note that the indices  $a, b$  are implicit in this notation, and that we also use it for mesonic operators, for which the paths are closed loops in the dimer. Let us denote  $\Sigma_{\mathcal{P}_\alpha}$  the 4-cycle that corresponds to one such operator by the AdS/CFT correspondence [30], as described in the previous section. In this section we argue that, considering the configuration of a single D3-brane at the CY singularity, the operator  $\mathcal{O}_{\mathcal{P}_\alpha}$  is precisely generated as (part of) the amplitude of an E3-brane instanton wrapped on  $\Sigma_{\mathcal{P}_\alpha}$ .

The appearance of  $\mathcal{O}_{\mathcal{P}_\alpha}$  in the instanton amplitude can be regarded as arising from the integration over fermion modes in the E3-D3 open string sector,  $\beta, \gamma$ , in the  $\bar{\square}_a, \square_b$ , respectively, with a coupling in the instanton world-volume action

$$\beta \mathcal{O}_{\mathcal{P}_\alpha} \gamma. \tag{4.1}$$

For mesonic operators, the modes  $\beta, \gamma$  form a vector-like pair. When  $\mathcal{O}_{\mathcal{P}}$  involves several bifundamental chiral multiplets, we refer to these couplings as “long”. The operator  $\mathcal{O}_{\mathcal{P}}$  takes zero vacuum expectation value exactly on the four cycle  $\Sigma_{\mathcal{P}_\alpha}$ , while it gives mass to the modes  $\beta, \gamma$  away from  $\Sigma_{\mathcal{P}_\alpha}$ . This fact is a consistency check that the coupling (4.1) is generated by an E3-brane instanton wrapped on  $\Sigma_{\mathcal{P}_\alpha}$ .

Notice that the complete structure of the instanton amplitude may contain additional insertions, due to extra fermion modes, etc, which actually depend on the details of the global compactification. As explained in the introduction, we center our analysis in this part of the instanton prefactor, which depends only on local properties of the configuration.

The simplest case in which we claim that our proposal holds is the case of a E3-brane wrapped in a single irreducible cycle, which we expect to be associated to an operator  $\mathcal{O}_{\mathcal{P}}$  which does not factorize. We expect this close relation between factorizability of the cycle and the operator to hold in general. Nevertheless, this is a somewhat subtle point, and we want to clarify it in the following.

As discussed at the end of section 3.3, in the case of a general toric singularity it is important to include the bundle data in the specification of the string dual to the baryon operator. When we speak of factorizability and recombability here, it is understood that the bundle should be taken into account. More simply, one could frame the discussion in the mirror manifold, as we will do in section 4.6.

Another issue is that, since in fact for  $N > 1$  any BPS operator can be factorized as a product of bifundamentals, we should clarify what happens for cycles which are reducible but can be recombined into one smooth irreducible cycle.<sup>7</sup> When  $\Sigma_{\mathcal{P}_\alpha}$  is reducible, our map

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<sup>6</sup>The name “single determinant” comes from the fact that in the case of general  $N$  such concatenated chains give operators that can be written as a single determinant of the chain of fields. As we will discuss in section 5, the set of single-determinant operators generate the whole set of BPS operators.

<sup>7</sup>A related issue for the case  $N > 1$  is that any operator of the form  $\det AB$  factorizes as  $\det A \det B$ .

implies that the corresponding operator is generated by a multi-instanton process, with one E3-brane wrapped on each component of the reducible 4-cycle (see [65] for instantons on reducible cycles, and [64, 66–68] for recent papers on multi-instantons). Multi-instantons imply additional zero modes, and the discussion of their 4d amplitude is more involved. Nevertheless, we argue that the general statement of the relation between 4-cycles  $\Sigma_{\mathcal{P}}$  and BPS operators  $\mathcal{O}_{\mathcal{P}}$  holds in general, by applying the following deformation argument. It is possible to regard the reducible cycles  $f = 0$ , e.g.  $xy = 0$  or  $x^2 = 0$ , as singular limits of irreducible 4-cycles  $f + \epsilon g = 0$  like  $xy - \epsilon = 0$  or  $x^2 - \epsilon y = 0$ . Note that here  $\epsilon$  should be regarded not as an instanton bosonic mode, to be integrated over, but rather as a tunable parameter fixed by boundary conditions, or the complex structure moduli of the global compactification.<sup>8</sup> The 4d amplitude of the irreducible instanton leads to the 4d operator  $f + \epsilon g$ , as in the above paragraphs. In the limit  $\epsilon \rightarrow 0$ , the instanton becomes reducible and seemingly more complicated. However, the dependence in  $\epsilon$  determined away from that point can be extended using holomorphy<sup>9</sup> of the 4d non-perturbative F-term in the (complex structure dependent) parameter  $\epsilon$ . Hence at  $\epsilon = 0$  it must reduce to just an insertion of  $f$  (despite the fact that the process generating this insertions may be rather involved).

Thus we expect our general arguments to apply even for reducible 4-cycles which admit a recombination into a single smooth one,  $\Sigma_{\mathcal{P}} = \Sigma_1 + \dots + \Sigma_K$ . The complete operator generated by the multi-instanton process defined by  $\Sigma_{\mathcal{P}}$  is given by the concatenation of the operators generated by the different instantons associated to the individual  $\Sigma_i$ . Clearly this implies that all “single determinant” operators of the  $N = 1$  theory, defined by a path of concatenated bifundamentals, can be regarded as generated by a recombinable multi-instanton of this kind. On the other hand, “multi determinant” operators of the  $N = 1$  theory, defined by products of the above, namely by several non-concatenable paths, correspond to multi-instanton processes which do not admit a recombination into a single smooth one. Correspondingly, these are indeed described by multi-particle states in AdS/CFT, arising from different wrapped D3-branes. We therefore focus on “single determinant” operators, since they contain all the essential information about the spectrum of BPS operators. We will apply the above considerations about recombinability when necessary, and even abuse language using couplings like (4.1) in such situations, and treating the process as a single-instanton one.

As mentioned above we will not provide a formal proof of the correspondence, but we will argue in several different ways for the existence of the couplings and zero modes that we require. Let us provide here a short summary of the arguments in the rest of this section.

Sections 4.2 and 4.3 review some results already known in the literature which support our viewpoint, for the particular cases of single field and mesonic operator insertions.

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<sup>8</sup>We force the recombination of the instantons by changing their complex structure. See [65] for a discussion of instanton recombination by motion over Kahler moduli space.

<sup>9</sup>See [64, 66] for a general discussion of holomorphy of non-perturbative superpotential and higher F-terms, and reducible instantons in loci in Kahler moduli space. Although here we are interested in the (much more holomorphic) discontinuity complex codimension one loci in the complex structure moduli space in the spectrum of BPS branes, the microscopic analysis for those systems could be carried out in a similar spirit for the systems at hand to show the continuity of the 4d contribution as  $\epsilon \rightarrow 0$ .

Section 4.4 argues that couplings of the form (4.1) are present for any orbifold singularity. The argument proceeds essentially by orbifolding the results known from flat space. Then, by using partial resolution, section 4.5 argues that such couplings are present for any toric singularity.

Section 4.6 gives an independent argument for the validity of our result. We show explicitly how we can find the disc worldsheet instantons giving the coupling (4.1) in some simple situations. This picture has the advantage that everything is geometrical (in particular, there are no subtleties having to do with Chan-Paton factors), but also has the drawback that the special Lagrangian cycles dual to the cycles wrapped by the instanton are not explicitly known. Nevertheless, we argue that with a reasonable ansatz for the topology of the dual cycles (based on the well-understood single insertion case), one obtains couplings of the form (4.1).

## 4.2 Single field insertions

The simplest BPS operators of the form described above in the  $N = 1$  theory are given by the bi-fundamental chiral multiplets  $\Phi_{ab}$  themselves. The correspondence between branes wrapped on 4-cycles and such bifundamentals has already been considered in appendices of [69] for D7-branes, and of [61] for E3-brane instantons. Indeed, using dimer diagrams it is straightforward to verify that to a given bi-fundamental multiplet one can associate a divisor in the singular geometry, such that an E3-brane wrapped in the latter has fermion zero modes  $\beta, \gamma$ , coupling as  $\beta \Phi_{ab} \gamma$ . In our present setup, we regard this result as the simplest realization of the correspondence between 4-cycles and BPS operators of the  $N = 1$  theory. In fact, it was already argued in these papers that this correspondence is exactly the same as that obtained from the AdS/CFT correspondence.

## 4.3 Mesonic operators

Let us argue that this correspondence applies also to mesonic operators, a discussion in fact related to systems studied in [70–72]. The consideration of mesonic operators will naturally provide us with examples of the correspondence beyond single field insertions.

Consider the simplest situation of a single D3-brane in flat transverse space  $\mathbb{C}^3$ , parametrized by  $(z_1, z_2, z_3)$ . Abusing notation, we also denote  $z_i$  the D3-brane adjoint chiral multiplets, parametrizing the D3-brane position. Consider an E3-brane instanton wrapped on the 4-cycle defined by e.g.  $z_1 = 0$ . In the E3-D3 open string sector there are fermion modes  $\beta, \gamma$ , with a world-volume coupling  $\beta z_1 \gamma$ , which reflects that the separation of the branes in  $z_1$  controls the mass of these modes. Thus, integration over these instanton fermion modes leads to an insertion of the mesonic operator  $z_1$ , similarly to the previous section. Notice that we manifestly recover the AdS/CFT map between the 4-cycle  $z_1 = 0$  and the BPS operator  $z_1$ .

This is just the E3-brane version of the result in [70] for non-perturbative effects of D7-branes in presence of D3-branes. It is also a particularly simple realization of the effect computed in [71].<sup>10</sup> In this paper, the authors considered the non-perturbative

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<sup>10</sup>Related results appear in [72], where the computations are done from the open string viewpoint. This is similar in spirit to our computations in this section.



superpotential generated by gaugino condensation on D7-branes wrapped on a non-compact 4-cycles, in a warped deformed conifold background, as a function of the location  $z_i$  of one D3-brane. The result involved a computation of the change of warped wrapped volume as a function of this position, leading to a modification of the instanton amplitude of the form (adapting already to E3-brane instantons rather than the fractional instantons involved in D7-brane gaugino condensates),

$$S = \int d^2\theta f(z_i) e^{-T} \tag{4.2}$$

where  $f(z_i) = 0$  is the equation of the 4-cycle wrapped by the instanton brane.

In fact, many of the ingredients of the configuration, like the 3-form fluxes, the complex deformation, or even the fact of being at a conifold, are actually not essential. The result has much more general validity, since it amounts to a computation in the closed string channel of the annulus diagram that corresponds to integrating over E3-D3 instanton fermion modes. Applied to our flat space example, the instanton wrapped on the 4-cycle  $z_1 = 0$  leads to the insertion of the (mesonic) operator  $z_1$ .

The argument applies to general singularities. Since a general mesonic operator correspond to a holomorphic function  $f(z_i)$  on the singular geometry, an instanton wrapped on the divisor  $f(z_i) = 0$  leads to a 4d effective vertex containing the mesonic operator  $f(z_i)$ . From the viewpoint of the instanton, this arises from integrating over E3-D3 fermion modes  $\beta, \gamma$ , with couplings  $\beta f(z_i) \gamma$ , reflecting that they become massive as the E3-brane is moved away from the D3-branes (namely, when the defining equation is modified to  $f(z_i) = \epsilon$ ). This shows the existence of general “long” couplings of the form (4.1), for mesonic operators.

For example consider a single D3-brane on a conifold described as  $xy - zw = 0$ , and an E3-brane wrapped on  $z = 0$ . In terms of the underlying D3-brane field theory, the coordinates are mesonic operators,

$$A_1 B_1 = x, A_2 B_2 = y, A_1 B_2 = w, A_2 B_1 = z \tag{4.3}$$

So the non-perturbative E3-brane instanton reads (assuming it generates a superpotential)

$$\int d^2\theta A_1 B_2 e^{-T} \tag{4.4}$$

Hence in general, for any given mesonic operator  $\mathcal{O}_{\mathcal{P}}$  of the  $N = 1$  theory there is a 4-cycle such that the wrapped E3-brane instanton leads to an insertion of  $\mathcal{O}_{\mathcal{P}}$  in the 4d effective action.

Notice the fact that the couplings of the form (4.1) involve the operator  $\mathcal{O}_{\mathcal{P}}$  modulo F-terms should be clear at this point. In fact, the rewriting of a mesonic operator in terms of the underlying fields is an operation which is defined modulo the F-term relations.

#### 4.4 Long baryonic couplings for orbifolds

We have argued that instantons can generate a variety of long couplings and BPS operators for some simple singularities. One simple way to show the appearance of long couplings in more general and more involved singularities is orbifolding. For instance, we may consider

orbifolds of  $\mathbb{C}^3$  by a discrete subgroup  $\Gamma$  of  $SU(3)$ , which we take to be abelian in order for the orbifold to admit a toric description. The gauge group splits as a product (maintaining the  $U(1)$  for momentary convenience) of  $U(1)^K$  with  $K$  the order of  $\Gamma$ , and each adjoint of the parent theory leads to a set of bi-fundamentals. For instance, considering the  $\mathbb{Z}_3$  orbifold generated by a rotation  $(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ , the three adjoints  $X, Y, Z$  lead to bifundamental fields  $X_{i,i+1}, Y_{i,i+1}, Z_{i,i+1}$ , transforming in the  $(\square_i, \bar{\square}_{i+1})$ . The superpotential of the theory is obtained by replacing the original adjoints in the parent superpotential by the bifundamentals they lead to, in all possible ways consistent with gauge invariance. Namely

$$W = X_{i,i+1}Y_{i+1,i+2}Z_{i+2,i} - X_{i,i+1}Z_{i+1,i+2}Y_{i+2,i} \quad (4.5)$$

In this orbifolding process, the fate of E3-brane instantons is easy to determine. In performing the quotient, one needs to specify the action of  $\Gamma$  on the Chan-Paton indices of open strings with endpoints on the E3-brane. The choice of this Chan-Paton phase, determines with which of the possible bi-fundamentals the corresponding E3-D3 fermion zero modes will couple in the quotient theory.

Consider a concrete example, corresponding to instantons leading to single field insertions. Consider an E3-brane defined by  $X = 0$  in the  $\mathbb{C}^3$  theory, thus leading to the insertion of the mesonic operator  $X$  in the 4d effective action. In performing e.g. the  $\mathbb{Z}_3$  quotient described above, there are three possible choices of Chan-Paton phase for the E3-brane. This phase enters in the orbifold projection on the E3-D3 fermion zero modes, and determines the coupling of the survivors to one of the three bifundamentals  $X_{12}, X_{23}, X_{31}$  in the quotient theory. Therefore each of the three possible E3-branes in the quotient theory lead to the insertion of one of these baryonic operators. One can operate similarly to obtain instantons with couplings to the other bifundamentals  $Y_{i,i+1}$  or  $Z_{i,i+1}$  in the quotient theory. Notice that single field insertions for orbifold theories already provide the simplest realization of the orbifolding procedure we are discussing in the present section.

Before continuing, let us make a few remarks on this simple example, which generalize to arbitrary orbifolds. First notice that orbifolding allows to deduce the appearance of baryonic operators from information on the appearance of mesonic operators. Notice also that this examples illustrates the above discussion on reducible vs recombinable cycles in section 4.1. Consider a system of three E3-branes, in the above  $\mathbb{C}^3/\mathbb{Z}_3$  example, each with one of the possible choices of Chan-Paton phase. The system of three E3-branes can recombine into a single dynamical E3-brane which can move away from the singularity. From our discussion in section 4.3, such E3-brane leads to the insertion of the mesonic operator whose vev parametrizes the E3-D3 distance. Indeed this agrees with our discussion of reducible 4-cycles which can recombine. The three different E3-branes lead to insertions of the operators  $X_{12}, X_{23}, X_{31}$ . Taken together, the multi-instanton process they generate leads to the insertion of their concatenation, namely the mesonic operator  $X_{12}X_{23}X_{31}$ , which in fact corresponds to the coordinate controlling the E3-D3 distance in the quotient theory.

Finally, the  $\mathbb{Z}_3$  orbifold singularity also illustrates an interesting feature in mapping the BPS operators under discussion, and the corresponding 4-cycles, with the E3-brane instantons. Indeed, the choice of Chan-Paton factor for a given 4-cycle can be described geometrically as the choice of a holonomy at infinity for the world-volume gauge field.

Equivalently, the 3-cycle defining the base of the conical 4-cycle is non-simply connected, and there is a discrete choice of Wilson line. This implies that in the AdS theory, for this 3-cycle there are different wrapped D3-brane states, which correspond to different baryons. This is nicely correlated with the existence, for such 4-cycle, of different E3-branes, coupling to different bifundamentals. This provides another nice piece of agreement between the E3-brane and the D3-brane viewpoint on BPS operators of the field theory.

Let us describe the extension of the above orbifolding procedure to operators involving several fields. Consider for instance the  $\mathbb{C}^3$  theory with an E3-brane leading to the operator  $X^2$ , which in fact corresponds to a system of two E3-branes recombining into a single one. Performing the quotient by the above  $\mathbb{Z}_3$  action, one needs to specify the Chan-Paton action on the E3-branes. Choosing the same Chan-Paton phase for both would lead to operators of the form e.g.  $X_{12}X_{12}$ , for which the two fields cannot be concatenated. This signals that the two E3-branes in the quotient cannot be recombined (the recombination parameter has been projected out by the quotient), hence it corresponds to an unavoidable genuine 2-instanton process. On the other hand, choosing different Chan-Paton phases leads to E3-branes generating insertions like e.g.  $X_{12}X_{23}$ , namely long baryonic operators. These systems correspond to E3-branes which admit a recombination into a single one, and work as an overall single-instanton process. As already mentioned, we focus on this kind of system, namely on E3-brane systems leading to concatenated chains of bifundamentals.

This construction generalizes easily to obtaining the orbifold descendants of general operators of the  $\mathbb{C}^3$  theory. For instance, operators like  $XY$  lead to operators  $X_{12}Y_{23}$ ,  $X_{23}Y_{31}$ ,  $X_{31}Y_{12}$ . The choice of Chan-Paton phase on the E3-brane system determines the endpoints of the chain of bifundamentals (namely the baryonic charges of the operator). The generalization should be clear. An important observation concerns the absence of ordering ambiguities thanks to the use of F-term relations. For instance, consider the operator  $Y^2Z^2$  in  $\mathbb{C}^3$ , and two of its possible descendants for a given choice of Chan-Paton action e.g.  $Y_{12}Y_{23}Z_{31}Z_{12}$  and  $Y_{12}Z_{23}Y_{31}Z_{12}$ . These turn out to be identical upon using the F-term equation for  $X_{12}$ , namely  $Y_{23}Z_{31} = Z_{23}Y_{31}$ , as obtained from (4.5).

#### 4.5 Long baryonic couplings for general singularities from partial resolution

In the previous section we have described the generation of long baryonic operators for orbifold theories by E3-brane instantons. Since partial resolutions of orbifold singularities can lead to non-orbifold singularities, we may follow the effects of partial resolution on E3-brane instantons in order to study long baryonic operators from E3-brane instantons in non-orbifold singularities. In fact, since any toric singularity can be regarded as the partial resolution of an orbifold singularity (of sufficiently large order), partial resolution can be used to obtain a general correspondence, for arbitrary toric singularities, between single determinant BPS operators and E3-branes on 4-cycles.<sup>11</sup> This correspondence is nicely correlated with the map between BPS operators and 4-cycles defined by the AdS/CFT correspondence.

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<sup>11</sup>In [73] the couplings of flavour D7 branes were studied in the T-dual brane tiling picture, and a subset of the “long” couplings we discuss in this section were argued to exist.

The main effects that a BPS operator (and the E3-brane instanton generating it) can suffer in a process of partial resolution are the following.

- All bifundamental fields in the chain defining the operator descend to fields in the resolved theory. The operator is unchanged and described by the same chain of fields in the resolved theory.
- One of the bifundamental fields in the chain gets a vev. The operator in the resolved theory is obtained by simply removing this bifundamental from the chain. Namely if the initial operator  $(\mathcal{O}_{\mathcal{P}})_{ab} = (\mathcal{O}_{\mathcal{P}_1})_{ac} \Phi_{cd} (\mathcal{O}_{\mathcal{P}_2})_{db}$ , with  $\Phi_{dc}$  getting a vev, the operator in the resolved theory is  $(\mathcal{O}'_{\mathcal{P}'})_{ab} = (\mathcal{O}_{\mathcal{P}_1})_{ac} (\mathcal{O}_{\mathcal{P}_2})_{db}$ . The operator remains “single-determinant” since the vev for  $\Phi_{cd}$  breaks of the two gauge factors  $c, d$  to the diagonal combination, so that the two sub-chains can be concatenated.
- One of the bifundamental fields in the chain becomes massive by superpotential couplings and is not present in the resolved theory. Consider the operator  $(\mathcal{O}_{\mathcal{P}})_{ab} = (\mathcal{O}_{\mathcal{P}_1})_{ac} \Phi_{cd} (\mathcal{O}_{\mathcal{P}_2})_{db}$ , with  $\Phi_{cd}$  becoming massive. To obtain the resolved theory the bifundamental is integrated out by using the F-term relations, which relate its value to some single determinant operator (possibly identically zero) say  $\Phi_{cd} = (\mathcal{O}_{\mathcal{P}_3})_{cd}$ , involving fields that survive in the resolved theory. Since the BPS operators generated by the instantons should be understood modulo F-terms, the resulting operator in the resolved theory is simply  $(\mathcal{O}'_{\mathcal{P}'})_{ab} = (\mathcal{O}_{\mathcal{P}_1})_{ac} (\mathcal{O}_{\mathcal{P}_3})_{cd} (\mathcal{O}_{\mathcal{P}_2})_{db}$ . This manifestly remains a single-determinant operator, i.e. a concatenated chain. In general, the replacement via F-term relation may require the replacement of a sub-chain in general longer than one bifundamental field.
- The above two operations act quite trivially on the E3-brane, which still passes through the singularity after the process. There is however a situation where this geometrical property of the E3-brane changes. Notice that in a process of partial resolution some baryonic charges disappear. This implies that some baryonic operators lose their non-trivial charges and become mesonic in the resolved theory. For an operator  $(\mathcal{O}_{\mathcal{P}})_{ab}$  this happen when the groups  $a, b$  are broken to the diagonal combination. The interpretation in terms of the E3-brane instanton is that the blowing-up process has grown a 2- or 4-cycle which separates the E3-branes from the D3-brane stack.

Let us discuss these main features by considering an illustrative example. Consider  $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ , with the orbifold generators associated to the twists  $(\frac{1}{2}, -\frac{1}{2}, 0)$  and  $(0, \frac{1}{2}, -\frac{1}{2})$ . The gauge group of the orbifold theory contains four factors, and the  $\mathbb{C}^3$  adjoints lead to the bifundamental fields  $X_{12}, X_{21}, X_{34}, X_{43}, Y_{23}, Y_{32}, Y_{14}, Y_{41},$  and  $Z_{31}, Z_{13}, Z_{24}, Z_{42}$ , in hopefully self-explanatory notation. The superpotential has the structure  $W \simeq XYZ - XZY$ , with indices distributed in all possible ways consistent with gauge invariance. The dimer diagram is shown in figure 1a. Figures b and c provide the partial resolution to the SPP and the conifold, which we are about to use, obtained by giving vevs to the fields  $X_{34}$  for the SPP, and to  $X_{34}, Z_{31}$  for the conifold. The dimer is a convenient tool to represent BPS operators, which correspond to paths joining two faces (which are the same

for mesonic operators), modulo homotopy deformations (F-term relations). The effects described above appear in this example as follows:

- The operator  $X_{43}X_{34}Y_{41}$  in the orbifold theory descends to the operator  $X_{43}Y_{41}$  in the SPP theory (which corresponds to a concatenated chain since the groups 3 and 4 become identified in the SPP theory).
- Consider the operator  $Z_{13}Y_{32}$  in the unresolved orbifold theory. The field  $Z_{13}$  ends up as a massive one in the resolution to SPP, as is manifest in the dimer, where it enters a bi-valent node. It is however simple to deform the path in the SPP dimer to obtain the operator  $X_{12}Z_{24}Y_{32}$  which is a concatenated chain (since 3 and 4 become identified) of fields massless in the SPP theory. This amounts to just using the F-term equivalence  $Z_{13}X_{34} = X_{12}Z_{24}$  in the unresolved orbifold theory and replacing  $X_{34}$  by its vev.
- It is easy to find baryonic operators of the unresolved orbifold theory which become mesonic upon losing its baryonic charges in the partial resolution. The simplest example is just  $X_{43}$ , which is a mesonic operator in the SPP theory.

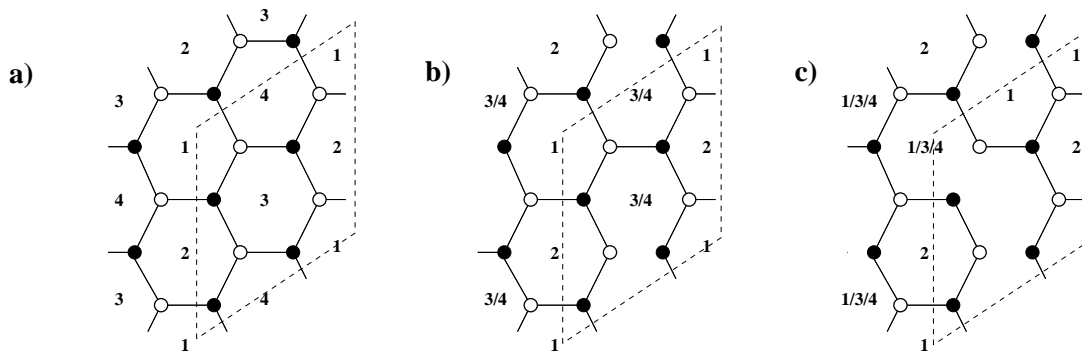
It is easy to realize that the realization of general toric singularities as partial resolution of orbifolds allows to reverse the above line of argument. Namely one can show that any BPS operator associated to a chain of bifundamentals in the non-orbifold theory can be regarded as the resolved version of a chain of bifundamentals in the orbifold theory. This construction produces the general map between arbitrary single determinant BPS operators for toric field theories and E3-branes instantons on 4-cycles producing them.

The correspondence can be easily argued to agree with the AdS/CFT map between operators and wrapped D3-branes, given that the chain of bifundamentals can be regarded as a monomial in the homogeneous variables of the symplectic quotient of the construction, which provide the defining equation for the 4-cycle on which to wrap the E3-brane. This is precisely the map used in the AdS/CFT context.

Finally, let us point out an interesting crosscheck allowed by partial resolution. Considering the conifold theory in figure 1c, it is possible to resolve it completely to  $\mathbb{C}^3$  by giving a vev to any of the bi-fundamentals. This partial resolution allows to recover long couplings in the  $\mathbb{C}^3$  theory by starting with long couplings of the conifold theory. We have thus closed the circle and obtained a consistent picture of all operators which can be generated using E3-brane instantons. Thus by orbifolding and partially resolving, one can reach the general result that any single determinant BPS operator can be generated from a suitable E3-brane instanton.

#### 4.6 The D6-brane mirror picture

We can provide further arguments in favour of the couplings previously discussed by using the mirror of the system of D3-branes at the singularity. These are described in appendix A.2, following [21]. The mirror  $\mathcal{W}$  of a toric Calabi-Yau variety  $\mathcal{M}$  can be obtained starting from its toric diagram [74–76], as follows. Assign complex coordinates  $x, y$  to the



**Figure 1.** The dimer diagram for the  $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$  theory (a) and its partial resolution to the SPP singularity (b) and the conifold (c). In order to keep the relation to the unresolved orbifold theory, we have not integrated out the bifundamental fields in di-valent nodes (mass terms in the superpotential).

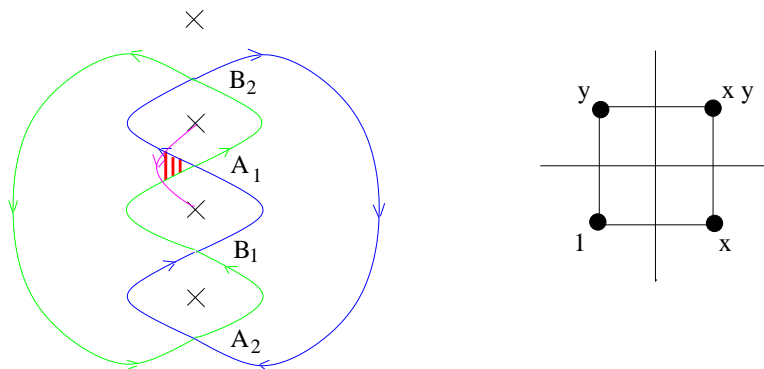
two axis of the toric diagram and associate a monomial  $x^i y^j$  to the point with coordinates  $(i, j)$  in the toric diagram. Define the polynomial  $P(x, y)$  as the sum of all these monomials with arbitrary complex coefficients.<sup>12</sup> The mirror variety  $\mathcal{W}$  is defined by the equation  $P(x, y) = uv$  where the coordinates  $u, v$  take values in  $\mathbb{C}^*$ . We can represent  $\mathcal{W}$  as a double fibration over the complex plane with coordinate  $z$

$$uv = z \quad , \quad P(x, y) = z. \quad (4.6)$$

The first equation describes a  $\mathbb{C}^*$  fibration, while the second equation describes a fibration of a Riemann surface. The structure of  $\mathcal{W}$  is essentially encoded in the latter fibration, and in particular on the fiber over  $z = 0$ . The Riemann surface  $P(x, y) = 0$  has genus equal to the number of internal points of the toric diagram and punctures corresponding to the external edges of the dual diagram  $\mathcal{C}^*$ . For example in the conifold case we have a Riemann surface that is topologically a sphere with four punctures, given by the defining equation:  $P(x, y) = 1 + x + y + xy$  (see figure 2). In this mirror geometry, the gauge D3-branes correspond to D6-branes wrapped on 3-cycles, which project on the Riemann surface to non-trivial 1-cycles wrapping non-trivially around the different punctures, in a way determined by the dimer diagram, see appendix A.2. Intersections of these 1-cycles support bifundamental chiral multiplets, while oriented disks defined by different 1-cycles support worldsheet instantons leading to superpotential couplings.

The mirror picture provides a nice geometric realization of the euclidean instanton branes, their charged fermion zero modes, and their couplings, as we now describe. The mirror picture of the E3-branes corresponds to E2-branes wrapped on non-compact special Lagrangian 3-cycles, which project as 1-cycles in the Riemann surface, escaping to infinity along two punctures. In fact, some of these non-compact 3-cycles have appeared (describing

<sup>12</sup>These complex coefficients parametrize the complex structure of the mirror manifold  $\mathcal{W}$ , and they are mapped to the Kahler structure parameters of  $\mathcal{M}$  under the mirror map. Their values are not relevant for our simplified discussion.



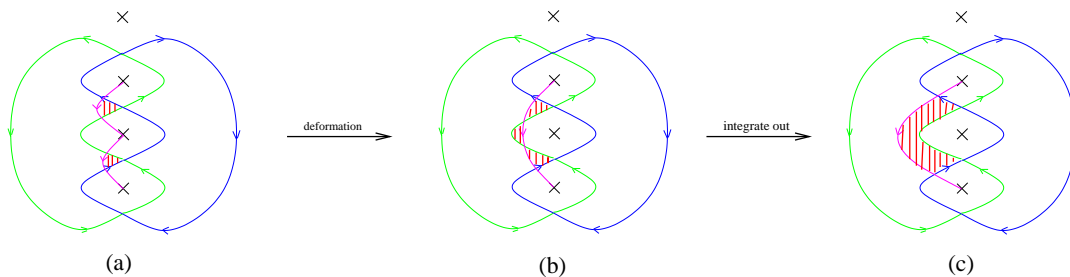
**Figure 2.** The mirror Riemann surface for the conifold, with punctures shown as crosses. The two 1-cycles in green and blue correspond to the two D3-brane gauge groups and the pink 1-cycle connecting the two punctures corresponds to the E3-brane instanton. The disk leading to the coupling  $\alpha A_1 \beta$  is painted in red stripes. The instanton amplitude thus produces an insertion of the field  $A_1$  (for the  $N = 1$  theory) in the D3-brane field theory. The right hand part of the figure shows the toric diagram for the conifold and its  $(p, q)$ -web.

the mirror of flavour D7-branes) in [69]. The intersection of the E3-brane non-compact 1-cycle with the D3-brane compact 1-cycles lead to charged fermion zero modes of the E3-brane instanton. Also, the disks bounded by a given E3-brane non-compact 1-cycle and the D3-brane compact 1-cycles in the Riemann surface support worldsheet instantons contributing to the couplings of the E3-brane instanton to a BPS operators in the 4d field theory.

The explicit map between holomorphic 4-cycles and special Lagrangian cycles is not know in general, thus in our analysis we consider a shortcut. We start with a basic set of non-compact 1-cycles, which correspond to E3-branes with fermion modes coupling to the basic bifundamental chiral multiplets. In addition, we construct more general E3-brane 1-cycles by combining basic 1-cycles which share a common puncture. The physical interpretation is that one can form bound states of the basic 1-cycles by giving vevs to fields in the E3-E3' open string sector, triggering recombination of cycles. The fermion zero modes and couplings of the resulting combined 1-cycle are manifest from the Riemann surface picture, and agree with the naive field theory analysis. Let us explain this procedure using the conifold example.

Consider the 1-cycles corresponding to the E3-brane instantons coupling to the elementary fields  $A_i, B_j$  in the conifold theory. As mentioned above, they are non-compact 1-cycles stretched between punctures, and defining suitable disks involving the corresponding bifundamental. The 1-cycles corresponding to E3-branes with the desired structure of fermion zero modes and couplings, namely  $\alpha A_i \beta, \alpha B_j \beta$ , are shown in figure 2. Note that the pink 1-cycle on the Riemann surface seems to define two disks, to its right and its left. However, only the disk on the right has a well-defined boundary orientation, and can really support a worldsheet instanton.

As discussed above, these basic cycles correspond to holomorphic 4-cycles (defined by



**Figure 3.** The two E2-brane instantons (a) can be recombined to a single E2-brane instanton (b) with coupling  $\alpha A_1 \gamma + \gamma \delta + \delta B_1 \beta$ . After integrating over the two modes  $\gamma, \delta$  the coupling in (b) is equivalent to  $\alpha A_1 B_1 \beta$ , as shown pictorially in (c). Integration over the remaining modes leads to the appearance of the mesonic BPS operator  $A_1 B_1$  in the instanton non-perturbative amplitude.

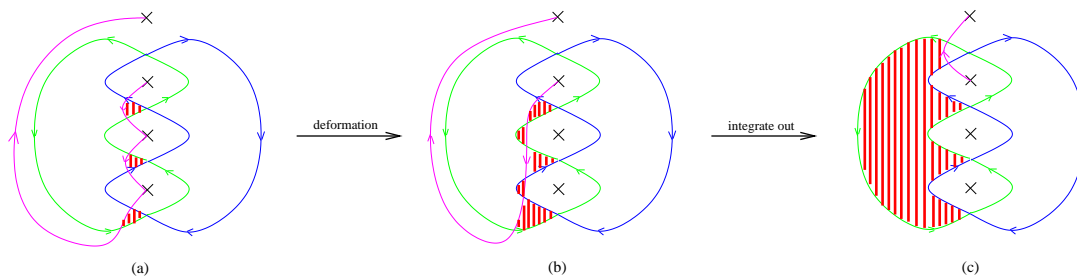
the equations  $x_r = 0$  in the homogeneous coordinates), and thus define supersymmetric 3-cycles in the mirror picture. Consider for instance the two basic 1-cycles giving rise to instantons coupling to  $A_1, B_1$ . In the type IIB picture, the two instantons correspond to two 4-cycles,  $x_1 = 0$  and  $x_2 = 0$ . This is a situation where we argued that the two-instanton process can be regarded as a limit of a one-instanton process, for a single E3-brane instanton wrapping the recombined 4-cycle  $x_1 x_2 + \epsilon = 0$ , in the limit  $\epsilon \rightarrow 0$ . Even in this limit, there is a non-trivial contribution of the instanton, leading to the insertion of the BPS operator  $A_1 B_1$ . We can now show that this construction is nicely reproduced using the mirror picture.

Consider the two basic E3-brane 1-cycles in figure 3a, describing instantons with fermion modes and couplings  $\alpha A_1 \gamma, \delta B_1 \beta$ . The two 1-cycles share a common puncture, corresponding to the fact that the IIB 4-cycles intersect over a complex curve. This intersection supports an E3-E3' mode  $\phi_{33'}$  (for whose existence we choose appropriate boundary conditions at infinity) with couplings  $\gamma \phi_{33'} \delta$ , which follows pictorially from a disk in the Riemann surface. A vacuum expectation value for this mode corresponds to the deformation parameter  $\epsilon$  mentioned above, leading to a single E3-brane bound state, whose recombined 1-cycle is shown in figure 3b. The triangle structure in the resulting picture lead to couplings  $\alpha A_1 \gamma + \gamma \delta + \delta B_1 \beta$ . One can thus integrate over the charged fermionic modes  $\gamma, \delta$  and obtain the coupling  $\alpha A_1 B_1 \beta$ . This corresponds pictorially to deforming the 1-cycle to figure 3c. Further integration over the remaining modes leads to the insertion of  $A_1 B_1$  operators in the 4d instanton amplitude.

In fact, even in the two-instanton process (with no recombination), one can use the couplings in figure 3a to saturate over the zero modes  $\gamma, \delta, \alpha, \beta$  and obtain the insertion of the operator  $A_1 B_1$  from the two-instanton process. In what follows, we will abuse language and use the above pictorial representation to discuss processes involving multi-instantons which can recombine into a single one, even when no actual recombination is implied. The procedure can be describe using two simple rules:

- Two instantons coming in and out of the same puncture can be recombined into a single instanton.





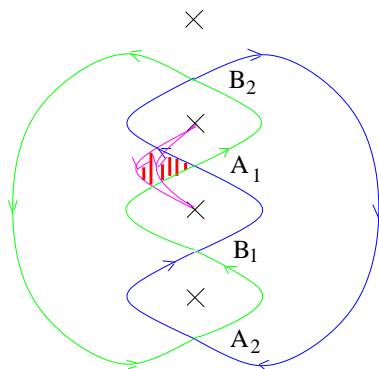
**Figure 4.** The three E2-brane instantons (a) can be recombined to a single E2-brane instanton (b) with coupling  $\alpha A_1 \gamma + \gamma \delta + \delta B_1 \mu + \mu \nu + \nu A_2 \beta$ . After integrating over the four modes  $\gamma, \delta, \mu, \nu$  the coupling in (b) is equivalent to  $\alpha A_1 B_1 A_2 \beta$ , as shown pictorially in (c). Integration over the remaining modes leads to the appearance of the baryonic BPS operator  $A_1 B_1 A_2$  in the instanton non-perturbative amplitude.

- One can deform 1-cycles to eliminate disks involving two intersections between the E3- and D3-brane 1-cycles (mass terms for non-chiral fermion modes). This correspond to integrating over the massive charged fermionic modes.

Consider a further example, leading to an instanton coupling to the operator  $A_1 B_1 A_2$ . The pictorial representation, according to the above rules, is shown in figure 4. The combined instanton system can be regarded as having the fermion modes and couplings  $\alpha A_1 \gamma + \gamma \delta + \delta B_1 \mu + \mu \nu + \nu A_2 \beta$ . Once we integrate over the four fermion modes  $\gamma, \delta, \mu, \nu$  we obtain the equivalent coupling  $\alpha A_1 B_1 A_2 \beta$  represented by the disk in the last figure. Integrating over these two charged zero modes give rise to the non perturbative insertion of the operator  $A_1 B_1 A_2$ .

It is important to underline that there are other situations where the multiple instantons behave as individual objects. These processes lead to many additional zero modes, coming e.g. from the individual goldstinos of the different instantons. Moreover, each instanton carries its set of charged fermion zero modes, and integration over them leads to the insertion of a BPS operator. Hence the multi-instanton process leads to a “multi-determinant” BPS operator in the field theory (and correspondingly, the boundary of the 4-cycles corresponds to a multi-particle set of D3-branes). This also has a nice interpretation in terms of the mirror geometry. E3-brane instantons which cannot form a bound state are described by 1-cycles which cannot recombine according to our above rules. Namely, they do not share a puncture, or they do not have correct orientations when they do. The structure of fermion modes and couplings from the Riemann surface automatically leads to the insertions of “multi-determinant” BPS operators.

Let us consider a simple example. Consider the two 1-cycles in figure 5. They describe two mutually BPS instantons, each of them coupling to  $A_1$ , which cannot be recombined (due to mismatch of orientations at the common punctures). In the IIB picture this corresponds to the embedding equation  $x_1^2 = 0$ . Namely two E3-brane instantons wrapped on the 4-cycle  $x_1$ . The system cannot form a bound state, since the equation  $x_1^2 = 0$  cannot be deformed into a single one in a way consistent with the  $\mathbf{C}^*$  quotient. From the mirror



**Figure 5.** The 1-cycles describing instantons with couplings  $\alpha_1 A_1 \beta_1 + \alpha_2 A_1 \beta_2$ . The two-instanton process leads to the insertion of the operator  $A_1^2$  for  $N = 1$ , or  $(\det(A_1))^2$  for general  $N$ .

picture, we see that the instantons have fermion modes and couplings  $\alpha_1 A_1 \beta_1 + \alpha_2 A_1 \beta_2$ . Integrating over the charged zero modes  $\alpha_i, \beta_j$  we have an insertion of the operator  $A_1^2$ . Since the bi-fundamental structure of  $A_1$  does not allow a concatenation of the two insertions, this corresponds to a “multi-determinant” operator. Equivalently, considering the theory for arbitrary  $N$ , the instantons generate the insertion of the operator  $(\det(A_1))^2$ .

### 5 BPS operators from E3-brane instantons: extension to $N$ D3-branes

In the previous section we have shown a remarkable relation between the E3-brane embeddings in  $C(H)$  and the BPS operators in the quiver theory of a single D3-brane at the singular point. Namely, once we specify the geometric embedding and the holonomy at infinity of the world volume gauge field, the E3 brane generates dynamically the corresponding BPS operator. In this section we discuss the map between E3-brane instantons and BPS operators for systems of  $N$  D3-branes at the toric singularity.

In passing from  $N = 1$  to general  $N$ , the spectrum of BPS operators becomes much more complicated, and in general the correspondence between BPS operators and string theory objects has to be studied at the level of a generating set of BPS operators. For instance, as explained in section 3.2 a generating set is provided by operators of the form (3.7), namely the gauge invariant  $N$  times symmetric products of concatenated chains of fields (analogous to the “single-determinant” operators of the  $N = 1$  case). The set of all BPS operators is obtained by taking products of these operators, and linear combinations thereof. Let us focus on BPS operators given by linear combination of operators of the form (3.7), to which we refer as ‘single-particle’ for the moment. In the AdS/CFT setup, such BPS operators correspond to D3-brane states in the Hilbert space of the quantum mechanics in the space parametrized by the coefficients  $\{c_P\}$  in the defining equations of the holomorphic 4-cycles. A particular basis of this Hilbert space is given by the states (3.9), dual to the operators (3.7).

In this section we show that E3-brane instantons provide, via the computation of the non-perturbative field theory operators they induce, a set of BPS operators which provide

a generating set for all ‘single particle’ BPS operators. Namely any ‘single-particle’ BPS operator can be described as a linear combination of the basis provided by the E3-brane instantons.<sup>13</sup> At the level of the AdS/CFT setup, the D3-brane states corresponding to the E3-brane instantons are those dual to determinant operators, as discussed in section 5.1, and provide a basis of the same Hilbert space spanned by the states (3.9), as shown in section 5.2. Therefore, although there is no one-to-one correspondence between BPS operators and E3-brane instantons, the E3-brane instantons do provide a generating set of BPS operators. This is enough to support the view that the correspondence between E3-brane instantons and BPS operators underlies the familiar one-to-one map between quantum D3-brane states and BPS operators.

### 5.1 The determinant operators

The considerations in section 4 for the  $N = 1$  case allow a simple generalization to arbitrary  $N$ . Using the couplings between E3-brane instantons and D3-branes in the  $N = 1$  case, we may increase the range of D3-brane Chan-Paton indices to obtain the worldvolume fermion modes and couplings of the general  $N$  system. Namely, for the operator  $\mathcal{O}_{\mathcal{P}}$  corresponding to any concatenated chain of bifundamentals described by a path  $\mathcal{P}$  in the dimer, there is an E3-instanton wrapped on a holomorphic 4-cycle with charged fermion modes and couplings  $\alpha\mathcal{O}_{\mathcal{P}}\beta$ . Here the modes  $\alpha$  and  $\beta$  transform in the  $\bar{\square}$  and  $\square$ , of the gauge groups where the path  $\mathcal{P}$  start and end, respectively (and which are the same for a mesonic operator). Integration over these fermion modes leads to insertion of the field theory BPS operator

$$\det \mathcal{O}_{\mathcal{P}} = \epsilon_{\tilde{p}_1, \dots, \tilde{p}_N} \epsilon_{k_1, \dots, k_N} (\mathcal{O}_{\mathcal{P}})_{p_1 \tilde{k}_1} \dots (\mathcal{O}_{\mathcal{P}})_{p_N \tilde{k}_N} . \tag{5.1}$$

Equivalently, for each possible 4-cycle, or equivalently for each choice of monomials  $\{c_{\mathcal{P}}\}$  in the defining equation, there is a BPS operator  $\det \mathcal{O}_{\mathcal{P}}$ .

The fact that this mechanism only generates determinant operators might suggest that such operators cannot generate the whole set of BPS operators, in particular operators of the form (3.7) with different entries  $\mathcal{O}_{\mathcal{P}_I}$ . However, the fact that we have an operator for each possible choice of 4-cycle (out of an infinite set, parametrized by the  $c_{\mathcal{P}}$ ) implies that the set of determinant operators generates the complete Hilbert space of baryonic operators, as we show in the next section. Note that the generating set of operators provided by these D3-brane states associated with E3 branes is unfamiliar from the CFT viewpoint, since it involves linear superpositions of operators with different conformal dimensions.

### 5.2 The space of general BPS operators and the Veronese map

Let us introduce the shorthand notation

$$(\mathcal{O}_{\mathcal{P}_1}, \dots, \mathcal{O}_{\mathcal{P}_N}) = \epsilon_{\tilde{p}_1, \dots, \tilde{p}_N} \epsilon_{k_1, \dots, k_N} (\mathcal{O}_{\mathcal{P}_1})_{p_1 \tilde{k}_1} \dots (\mathcal{O}_{\mathcal{P}_N})_{p_N \tilde{k}_N} \tag{5.2}$$

The set of operators for all possible choices of paths  $\{\mathcal{P}_I\}$  forms a basis of BPS operators. Let us consider the question of whether it is possible to reproduce the above operators by considering linear combinations of the determinant operators (5.1).

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<sup>13</sup>Note that we do not imply that one can take linear combinations of E3-brane instantons to achieve an arbitrary BPS operator in the field theory.

Consider first the simple example of the conifold with just two colours  $N = 2$ , and the reduced problem of constructing all the possible operators of baryonic charge 2 involving just by the two chains of bifundamentals  $A, ABA$ . One basis for these operators is given by

$$e_1 = (A, A) , e_2 = (ABA, ABA) , e_3 = (A, ABA) \tag{5.3}$$

and the general operator corresponds to a linear combination thereof. We denote the set of these by  $\mathcal{M}_2$ .

The operators  $A, ABA$  correspond to two specific monomials of the homogeneous coordinates of the conifold, that for simplicity we just call  $x, y$ :  $A \rightarrow x, ABA \rightarrow y$ . These monomials define sections of a non-trivial line bundle over the conifold. The generic section spanned by them is  $f = ax + by$ , where  $a, b$  are complex coefficients, in the defining equation of the 4-cycle  $f = 0$ , dual to the corresponding operator. Consider the determinant operator generated by an E3-brane instanton wrapped on this 4-cycle, namely

$$\begin{aligned} O(a, b) = \det(f) = (f, f) &= a^2(x, x) + 2ab(x, y) + b^2(y, y) \\ &= a^2(A, A) + 2ab(A, ABA) + b^2(ABA, ABA) \end{aligned} \tag{5.4}$$

Hence E3-brane instantons lead to operators  $O(a, b)$  for arbitrary choices of  $a, b$ . In order to show that this set is generating, we need to show that one can choose particular values of  $(a, b)$  to obtain three linearly independent operators generating  $\mathcal{M}_2$ . In this case it is easy to find that e.g.

$$l_1 = O(1, 0) = e_1 , l_2 = O(0, 1) = e_2 , l_3 = O(1, 1) = e_1 + e_2 + 2e_3 \tag{5.5}$$

provide a basis of the same space of operators  $\mathcal{M}_2$ .

In order to generalize the above construction to arbitrary  $N$ , it is convenient to express it in more geometric terms. Since the equation  $f = ax + by = 0$  is invariant under complex rescalings of  $a, b$ , the set of such equations is a  $\mathbb{P}^1$ , with homogeneous coordinates  $a, b$ . Similarly, the BPS operators are given by linear combinations of the  $e_i, \sum z_i e_i$ , up to overall rescaling of the  $z_i$ , namely they are parametrized by a  $\mathbb{P}^2$ . The computation of the BPS operator corresponding to an E3-brane instanton on a 4-cycle defined by  $(a, b)$  defines a map

$$\begin{aligned} v_1 : \mathbb{P}^1 &\rightarrow \mathbb{P}^2 \\ [a; b] &\rightarrow [a^2; ab; b^2] \end{aligned} \tag{5.6}$$

This is an example of a well know construction in algebraic geometry called the (degree 2) Veronese embedding. The image set in  $\mathbb{P}^2$  is given by the degree 2 curve

$$z_1 z_3 - z_2^2 = 0. \tag{5.7}$$

The set of operators  $O(a, b)$  will form a basis of  $\mathcal{M}_2$  if there exist at least three points in  $\mathbb{P}^1$  such that the vectors  $l(a, b) = a^2 e_1 + ab e_2 + b^2 e_3$  form a basis of  $\mathbb{C}^3$ . In geometric terms, a basis will not be obtained only if the image  $v_1(\mathbb{P}^1)$  is contained in a hyperplane

in  $\mathbb{P}^2$ . It is a familiar result of algebraic geometry that the Veronese curve is indeed not contained in any hyperplane.<sup>14</sup> Hence the set of operators  $O(a, b)$  forms a generating set.

Let us pass on to the general case. Using the tools and the intuition we have just developed we can show that the set of BPS operators induced by all possible E3-brane instantons form a generating set of all BPS operators of the quiver gauge theory for general  $N$ .<sup>15</sup>

We start the discussion explaining the general form of the Veronese map, which plays a prominent role in the argument, and which is a simple generalization of the  $N = 2$  discussion above. The general Veronese map is an embedding of  $\mathbb{P}^m$  in  $\mathbb{P}^n$  defined as follows. Consider  $\mathbb{P}^m$  parametrized by homogeneous coordinates  $[u_0, \dots, u_m]$ . The set of degree  $N$  homogeneous polynomials in these coordinates

$$\sum_{i_0 + \dots + i_m = N} w_{i_0 \dots i_m} u_0^{i_0} \dots u_m^{i_m} \tag{5.8}$$

defines a vector space of dimension  $\binom{m+N}{N}$ , with coordinates  $w_{i_0 \dots i_m}$ . Let us take as the target of our Veronese map  $\mathbb{P}^n$ , with  $n = \binom{m+N}{N} - 1$ . This parametrizes, as above, the set of homogeneous polynomials modulo an overall rescaling. The degree  $N$  Veronese map is obtained by considering the  $N^{\text{th}}$  power of a general monomial in the  $u_k$ , namely it is defined by the map

$$v_N : \mathbb{P}^m \rightarrow \mathbb{P}^n$$

$$w_{i_0 \dots i_m} = u_0^{i_0} \dots u_m^{i_m} \tag{5.9}$$

for  $i_0 + \dots + i_m = N$ . The resulting Veronese variety  $v_N(\mathbb{P}^m) \in \mathbb{P}^n$  can also be described by the following set of quadrics, which follow from the specific form of the embedding:

$$w_{i_0 \dots i_m} w_{j_0 \dots j_m} = w_{k_0 \dots k_m} w_{l_0 \dots l_m} \tag{5.10}$$

whenever  $i_0 + j_0 = k_0 + l_0, \dots, i_m + j_m = k_m + l_m$ . It is a general result that the variety  $v_N(\mathbb{P}^m) \in \mathbb{P}^n$  is not contained in any linear subspace of  $\mathbb{P}^n$ .

The application of this result to our problem of mapping of BPS operators for the conifold case should be clear by now. In fact, it can be used to solve the mapping problem for arbitrary toric singularities, as we now argue. A generic toric variety can be described as a symplectic quotient of  $\mathbb{C}^d$  by the action of an abelian group  $K$  ( $K \sim (\mathbb{C}^*)^{d-3} \times \Delta$  where  $\Delta$  is some abelian discrete group). Denoting  $x_1, \dots, x_d$  the homogeneous coordinates, the supersymmetric 4-cycles on which one can wrap E3-brane instantons are given by equations

$$f = \sum_{i_1, \dots, i_d} c_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d} = 0 \tag{5.11}$$

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<sup>14</sup>In this simple case the statement can be easily seen to be true, since it amounts to the trivial fact that it is not possible to rewrite the quadratic equation (5.7) into a linear equation of the form  $\sum c_i z_i = 0$ , for any constant  $c_i$ .

<sup>15</sup>As explained in the previous section, we need to consider processes involving multiple instantons, and they correspond to multi-particle D3-brane states in AdS/CFT. For simplicity we restrict to single-particle BPS operators, since they can generate the complete set of all BPS operators.

that transform homogeneously under  $K$ . Let us momentarily restrict the infinite set of coefficients  $a_P = c_{i_1 \dots i_d}$  to a finite set of  $m + 1$ . Then the set of holomorphic 4-cycles parametrizes a  $\mathbb{P}^m$  with homogeneous coordinates  $[\{a_P\}] = [a_1, \dots, a_{m+1}]$ . As discussed in section 4, every monomial  $x_1^{i_1} \dots x_d^{i_d}$  is associated to a concatenated chain of bifundamentals in the quiver field theory,<sup>16</sup> defining an operator  $\mathcal{O}_P$ . Increasing the range of Chan-Paton indices to general  $N$ , an E3-brane wrapped on the holomorphic 4-cycle leads to fermion zero modes and couplings  $\alpha \mathcal{O}_P \beta$ . Integration over fermion zero modes leads to the BPS operator  $\det \mathcal{O}_P = (\mathcal{O}_P, \dots, \mathcal{O}_P)$  in the 4d field theory. Expanding this determinant, i.e. taking all possible degree  $N$  products of the monomials contained in  $f$  (or rather its field theory translation), we obtain a linear combination of the set of operators of the form (3.7). In this way the set of operators obtained by all possible embeddings of the instanton is described by the degree  $N$  Veronese embedding from  $\mathbb{P}^m$  to  $\mathbb{P}^n$ , and we have argued above that such an embedding spans a base of all possible operators.

In order to complete the argument we just need to remove the cutoff  $m$ , a step which does not modify the conclusions.

## 6 Conclusions and outlook

In this paper we have discussed the field theory operators on the worldvolume theory of systems of D3-branes at toric singularities induced by E3-brane instantons wrapped on holomorphic 4-cycles on the Calabi-Yau geometry. We have argued that the resulting correspondence between E3-branes on 4-cycles and BPS baryonic operators in the quiver theory underlies and explains the AdS/CFT correspondence between wrapped D3-brane states on AdS and BPS operators on the boundary theory. Let us suggest some further applications and possible future research directions.

We have described the correspondence between E3-brane instantons and BPS operators in terms of a generating set of the latter. Namely any BPS operator can be written as a combination of the BPS operators directly induced by E3-brane instantons. This operation has a well-defined meaning in the AdS/CFT context, where the wrapped D3-branes from the E3-brane instantons form a complete set of quantum states of the Hilbert space dual to the set of BPS operators. Since the operation of taking linear combination has a physical meaning for the wrapped D3-brane states, there is a one-to-one map between wrapped D3-branes and BPS operators.

It would be interesting to explore physical realizations of this map at the level of the E3-brane instantons. One tantalizing possibility, suggested by the structure of the operators (3.7) and its dual states (3.9), is considering fractional instantons. In gauge theories, fractional instantons are physical objects whose action and number of fermion zero modes is a (typically  $1/N$ ) fraction of those for a genuine instanton. They have been suggested (see e.g. [77]) as responsible for the gaugino condensate of  $SU(N)$  super-Yang-Mills (or more generally for the non-perturbative superpotential of SQCD for  $N_f < N_c - 1$ ). They have also been proposed to play a prominent role in the strong coupling dynamics of more general supersymmetric gauge theories. Although the physical interpretation of fractional

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<sup>16</sup>Here we are simplifying slightly, and restricting ourselves to the single particle case.

non-gauge D-brane instantons it far from clear, it is tempting to propose that a genuine E3-brane instanton is made up of  $N$  fractional instantons, each coupling to a particular concatenated chain of bifundamentals along a dimer path  $\mathcal{P}$ . In such interpretation, the BPS operator (3.7) would correspond to a set of  $N$  fractional E3-brane instantons, each coupling to a different path  $\mathcal{P}_I$ ,  $I = 1, \dots, N$ . We leave this as an open direction for further research.

A second interesting tool to attempt the formalization of a one-to-one map between BPS operators and E3-branes is provided by the master space of the supersymmetric quiver theory introduced in [36, 37]. It describes the set of gauge invariant BPS operators of the D3-brane field theory, and has a systematic algebraic geometry description for the case of field theories on D3-branes at singularities. Moreover, it helps in reducing the problem of counting operators for general  $N$  to a plethystic exponentiation of the counting problem for the  $N = 1$  theory. Since for the  $N = 1$  theory there exists a nice explicit map between BPS operators and E3-brane instantons, it sounds plausible that the master space can provide a meaningful physical one-to-one map between BPS operators and E3-branes for general  $N$ .

Finally we would like to emphasize that our discussions of the field theory operators induced by general E3-brane instantons should have interesting applications to model building. Indeed, in the construction of realistic particle models on D3-branes at singularities it is natural to look for potential sources for particular interesting field theory operators, which are forbidden in perturbation theory and could be generated by instantons. Our tools can be used to propose precisely the kind of E3-brane instanton required for a given field theory operator. Thus our work allows for a broad generalization of the work in [61]. Another recent model building proposal related to our work is D-brane instanton mediated supersymmetry breaking in [78].

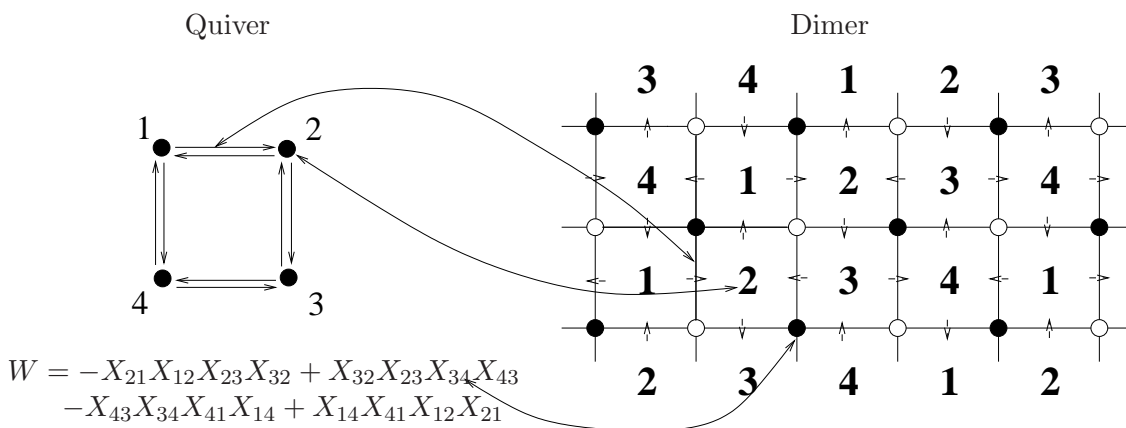
We expect much progress on the systematic understanding of E3-brane instantons and their corresponding BPS operators for these and other new directions.

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## A Short review of dimer models

In this appendix we would like to present a short review of the main aspects of dimer model technology that enter in our analysis above. For a more in-depth discussion, the reader is advised to consult the excellent reviews [22, 23].



**Figure 6.** Quiver and dimer for a  $\mathbb{Z}_2$  orbifold of the conifold. Faces in the dimer correspond to gauge groups, edges correspond to bifundamentals and each vertex corresponds to a superpotential term. Edges have an orientation determined by the coloring of the adjacent nodes.

### A.1 Quiver gauge theories and dimer diagrams

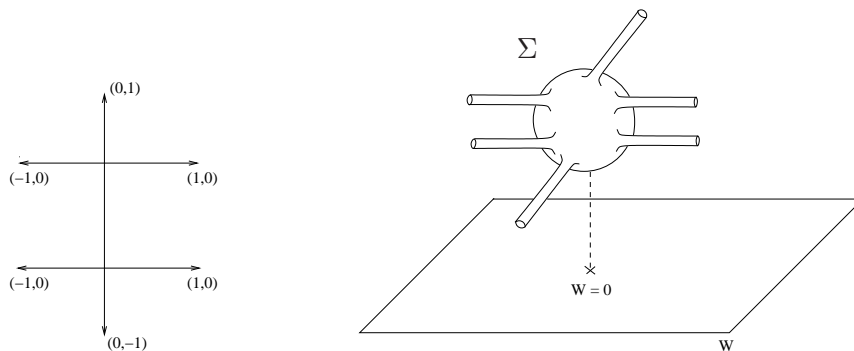
The gauge theory of D3-branes probing toric threefold singularities is determined by a set of unitary gauge factors (of equal rank in the absence of fractional branes, which we do not consider for the moment), chiral multiplets in bi-fundamental representations, and a superpotential given by a sum of traces of products of such bi-fundamental fields. The gauge group and matter content of such gauge theories can be encoded in a quiver diagram, such as that shown in figure 6a, with nodes corresponding to gauge factors, and arrows to bi-fundamentals. The superpotential terms correspond to closed loops of arrows, but the quiver does not fully encode the superpotential.

Recently it has been shown that all the gauge theory information, including the gauge group, the matter content and the superpotential, can be encoded in a so-called brane tiling or dimer graph [18, 19].<sup>17</sup> This is a tiling of  $\mathbb{T}^2$  defined by a bi-partite graph, namely one whose nodes can be colored black and white, with no edges connecting nodes of the same color. The dictionary associates faces in the dimer diagram to gauge factors in the field theory, edges with bi-fundamental fields (fields in the adjoint in the case that the same face is at both sides of the edge), and nodes with superpotential terms. The bi-partite character of the diagram is important in that it defines an orientation for edges (e.g. from black to white nodes), which determines the chirality of the bi-fundamental fields. Also, the color of a node determines the sign of the corresponding superpotential term.

The explicit mapping between this bipartite graph and the gauge theory, is illustrated in one example in figure 6.

<sup>17</sup>The brane tiling / dimer diagram can be dualized to an improved quiver diagram, the periodic quiver, which also encodes all this information.





**Figure 7.** a) An example of a web diagram (for the complex cone over  $\mathbb{F}_0$ ); b) the corresponding Riemann surface  $\Sigma$  in the mirror geometry.

### A.2 Dimer diagrams and the mirror Riemann surface

In [21] it was shown how, using mirror symmetry, it is possible to relate in a useful and explicit way the gauge theory on the D3 branes on the singularity and dimer models. Let us summarize the results of that paper here. The mirror geometry to a Calabi-Yau singularity  $\mathcal{M}$  is specified by a double fibration over the complex plane  $W$  given by

$$W = P(z, w) \tag{A.1}$$

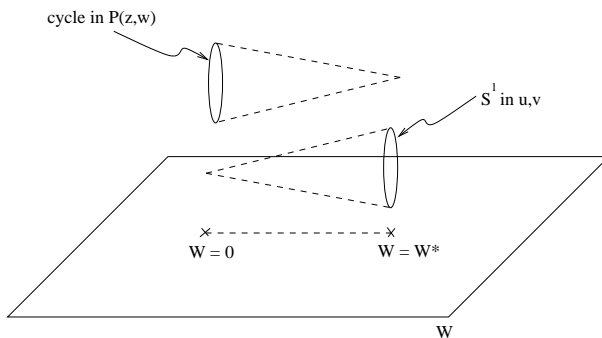
$$W = uv \tag{A.2}$$

with  $w, z \in \mathbb{C}^*$  and  $u, v \in \mathbb{C}$ . Here  $P(z, w)$  is the Newton polynomial of the toric diagram of  $\mathcal{M}$ . The surface  $W = P(z, w)$  describes a genus  $g$  Riemann surface  $\Sigma_W$  with punctures, fibered over  $W$ . The genus  $g$  equals the number of internal points of the toric diagram. The fiber over  $W = 0$ , denoted simply  $\Sigma$ , will be important for our purposes. It corresponds to a smooth Riemann surface which can be thought of as a thickening of the web diagram [74–76] dual to the toric diagram, see figure 7.

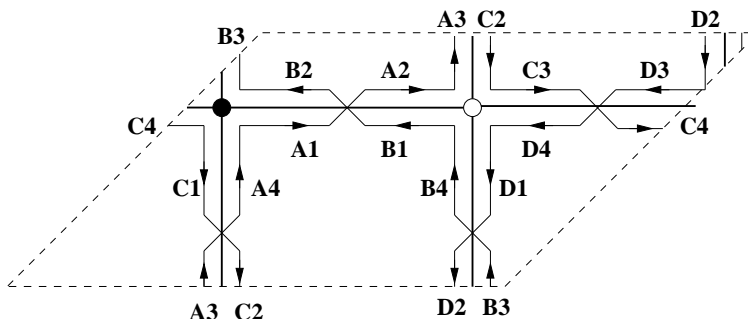
At critical points  $W = W^*$ , a cycle in  $\Sigma_W$  degenerates and pinches off. Also, at  $W = 0$  the  $S^1$  in  $W = uv$  degenerates. One can use these degenerations to construct non-trivial 3-cycles in the mirror geometry as follows. Consider the segment in the  $W$ -plane which joins  $W = 0$  with one of the critical points  $W = W^*$ , and fiber over it the  $S^1$  in  $W = uv$  times the 1-cycle in  $\Sigma_W$  degenerating at  $W = W^*$ , see figure 8. The result is a 3-cycle with an  $S^3$  topology. The number of such degenerations of  $\Sigma_W$ , and hence the number of such 3-cycles, is given by twice the area of the toric diagram.

Mirror symmetry specifies that the different gauge factors on the D3-branes in the original singularity arise from D6-branes wrapping the different 3-cycles. The 3-cycles on which the D6-branes wrap intersect over  $W = 0$ , precisely at the intersection points of the 1-cycles in  $\Sigma_{W=0}$ . Open strings at such intersections lead to the chiral bi-fundamental fields. Moreover, disks in  $\Sigma$  bounded by pieces of different 1-cycles lead to superpotential terms generated by world-sheet instantons.

Hence, the structure of the 3-cycles, and hence of the gauge theory, is determined by the 1-cycles in the fiber  $\Sigma$  over  $W = 0$ . This structure, which is naturally embedded in a



**Figure 8.** Structure of the non-trivial 3-cycles in the geometry  $\mathcal{W}$ . They are constructed by fibering over the segment joining  $W = 0$  and  $W = W^*$ , the  $S^1$  in the  $uv$  fiber (degenerating at  $W = 0$ ) times the 1-cycle in the  $P(z, w)$  fiber degenerating at  $W = W^*$ .

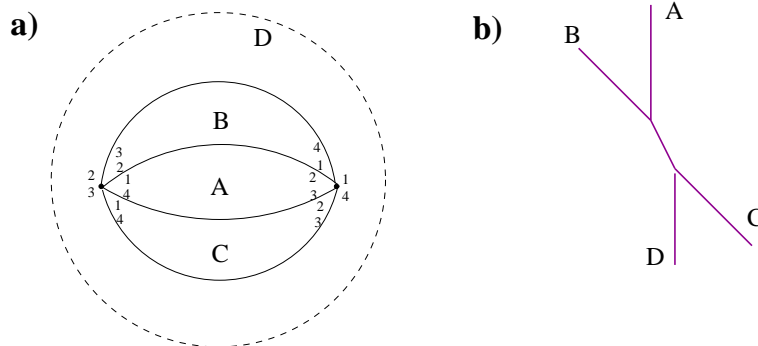


**Figure 9.** Dimer of the conifold with the corresponding zig-zag paths.

$\mathbb{T}^3$  (from the  $\mathbb{T}^3$  fibration structure of the mirror geometry), admits a natural projection to a  $\mathbb{T}^2$ , upon which the 1-cycles end up providing a tiling of  $\mathbb{T}^2$  by a bi-partite graph, which is precisely the dimer diagram of the gauge theory.

This last process is perhaps better understood (and of more practical use) by recovering the Riemann surface  $\Sigma$  from the dimer diagram of the gauge theory, as follows. Given a dimer diagram, one can define zig-zag paths (these, along with the related rhombi paths, were introduced in the mathematical literature on dimers in [79, 80], and applied to the quiver gauge theory context in [20]), as paths composed of edges, and which turn maximally to the right at e.g. black nodes and maximally to the left at white nodes. They can be conveniently shown as oriented lines that cross once at each edge and turn at each vertex, as shown in figure 9.

Notice that at each edge two zig-zag paths must have opposite orientations. For dimer models describing toric gauge theories, these zig-zag paths never intersect themselves and they form closed loops wrapping  $(p, q)$  cycles on the  $\mathbb{T}^2$ . This is shown for the conifold in figure 9 where the zig-zag paths A, B, C and D have charges  $(0, 1)$ ,  $(-1, 1)$ ,  $(1, -1)$ ,  $(0, -1)$  respectively.



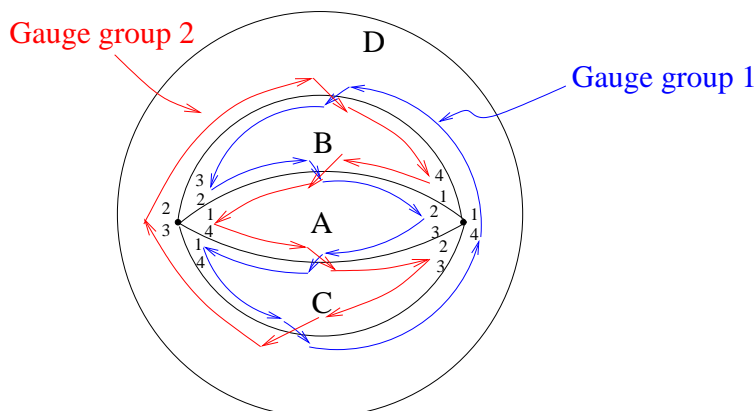
**Figure 10.** a) Tiling of the Riemann surface (which is topologically a sphere, shown as the complex plane) for the case of D3-branes at a conifold singularity. b) The web diagram, providing a skeleton of the Riemann surface, with asymptotic legs corresponding to punctures (and hence to faces of the tiling of  $\Sigma$ , and zig-zag paths of the original dimer diagram).

As shown in [21], the zig-zag paths of the dimer diagram associated to D3-branes at a singularity lead to a tiling of the Riemann surface  $\Sigma$  in the mirror geometry. Specifically, each zig-zag path encloses a face of the tiling of  $\Sigma$  which includes a puncture, and the  $(p, q)$  charge of the associated leg in the web diagram is the  $(p, q)$  homology charge of the zig-zag path in the  $\mathbb{T}^2$ . The touching of two of these faces in the tiling of  $\Sigma$  corresponds to the coincidence of the corresponding zig-zag paths along an edge of the dimer diagram. The tiling of  $\Sigma$  for the conifold is shown in figure 10a, while the corresponding web diagram is shown in figure 10b.

The dimer diagram moreover encodes the 1-cycles in the mirror Riemann surface, associated to the different gauge factors in the gauge theory. Consider a gauge factor associated to a face in the dimer diagram. One can consider the ordered sequence of zig-zag path pieces that appear on the interior side of the edges enclosing this face. By following these pieces in the tiling of  $\Sigma$  one obtains a non-trivial 1-cycle in  $\Sigma$  which corresponds precisely to that used to define the 3-cycle wrapped by the mirror D6-branes carrying that gauge factor. Using this map, it is possible to verify all dimer diagram rules (edges are bi-fundamentals, nodes are superpotential terms) mentioned at the beginning. The non-trivial 1-cycles in the mirror Riemann surface for the case of the conifold are shown in figure 11.

### A.3 Resolution of the singularity

In the discussion in the main text, we are particularly interested in seeing how does resolution of the singularity appear on the dimer model description. This resolution is expected to be represented in terms of the gauge theory as a Higgsing of some fields, in such a way that the low energy theory after Higgsing is the gauge theory of branes placed at the two daughter singularities. As shown in [81], it is possible to give a simple and beautiful recipe for understanding which fields get vevs using the mirror description of the system. Let us summarize the main points of the procedure here.



**Figure 11.** Tiling of the Riemann surface for the case of D3-branes at a conifold singularity, with the 1-cycles corresponding to the two gauge factors (shown as zig-zag paths of the tiling of  $\Sigma$ ).

In terms of the web diagrams, resolving the singularity corresponds to giving a finite length to one of the interior segments, representing a blowup of a  $\mathbb{P}^1$  in the toric geometry. The close relation between the Riemann surface  $\Sigma$  and the toric diagram suggests a way of reading the effect in the dimer, and hence in the gauge theory, of the blowup.

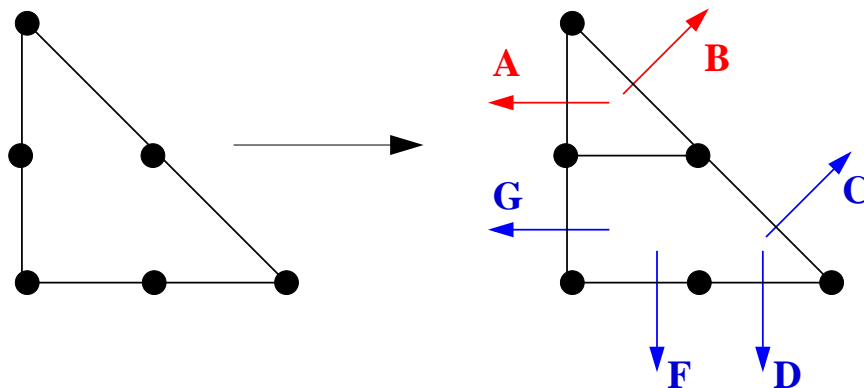
The basic idea is that one should identify which external legs of the web diagram go to which daughter singularity after the resolution. In the dimer this divides the zig-zag paths into two sets, since we have a one to one map between zig-zag paths and external legs of the web diagram of the toric singularity. Let us call the zig-zag paths in the first set paths of type 1 and those in the second set paths of type 2. In turn, since each edge in the dimer model is crossed by exactly two zig-zag paths, this divides the set of edges into three, namely those where two zig-zags of type 1 meet, those where two of type 2 meet, and those where zig-zag paths of mixed type meet. Let us denote this as edges of type 1, 2 and 3 respectively.

In terms of the mirror surface, a resolution consists of sending a given set of external legs to infinity (actually finite distance, but we will be integrating out the corresponding massive mediators). Then the theory divides into two sectors, one corresponding to each daughter singularity. In terms of dimer diagrams, what we have is that the original dimer diagram decomposes into two daughter diagrams, that we can call 1 and 2. The subdimer 1 is obtained from the original dimer diagram by removing all edges of type 2, and similarly for the subdimers of type 2. Edges of type 3 remain in both diagrams. This can be seen quite intuitively from the mirror, since edges of type 1 and 2 are localized in different sides of the resolution, while edges of type 3 can communicate with both sectors.

In terms of the gauge theory, what we have done is a Higgsing of the original theory, where we assign the following vacuum expectation values to bifundamentals:

$$\Phi_1 = \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\text{A.3})$$

where  $\Phi_i$  denotes the vev for the fields of type  $i$ . We see that these vevs force us to introduce



**Figure 12.**  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow SPP$  resolution in terms of the toric diagram. We have also drawn superposed, in color, the external legs of the web diagram. The desired resolution divides this set of external legs into A,B (in red) corresponding to a smooth geometry, and the rest of legs (in blue), which describe the SPP daughter singularity.

Fayet-Iliopoulos terms in order to cancel the D-terms and remain in a supersymmetric vacuum. Also, they trigger the recombination of some gauge factors into their diagonal combinations, which in terms of the dimer is represented as the recombination of the two faces adjacent to the edge that gets a vev.

Let us show how the procedure works for the  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow SPP$  resolution considered in the text.<sup>18</sup> The description of the resolution we are after in terms of toric diagrams is shown in figure 12.

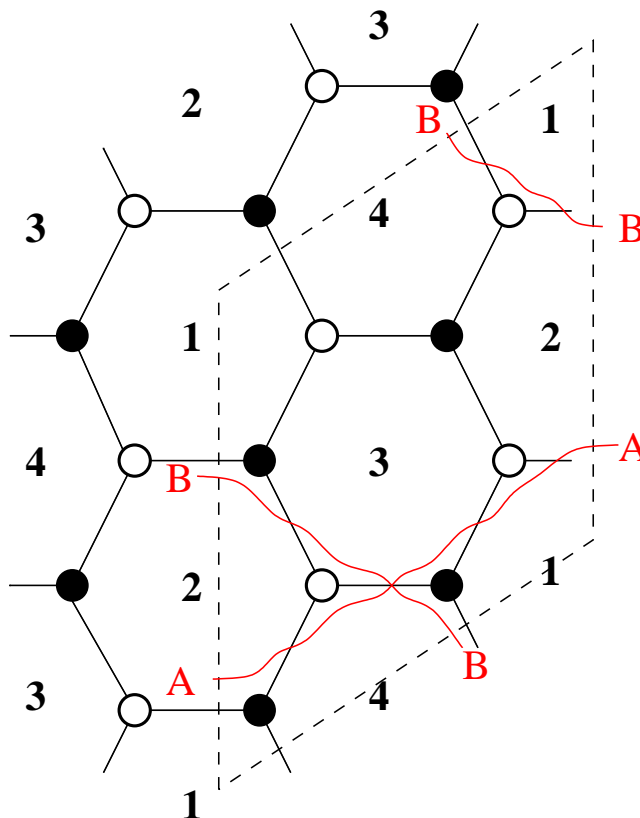
According to the general procedure outlined in this section, finding which field gets a vev in this resolution (in order to describe the SPP side) is just a matter of finding out on which edge A and B intersect.<sup>19</sup> This edge is localized on the flat space daughter “singularity”, and disappears from the SPP theory. The relevant zig-zag paths are shown in figure 13. They intersect over  $X_{34}$ , and thus this is the field that gets a vev on the SPP side.

Proceeding on a similar fashion with the C,D zig-zag paths (for example) one could higgs down the theory to the conifold, as we have done in the text.

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<sup>18</sup>We do not really need the technology described in this section in order to study this resolution, simply by giving a vev to any bifundamental of the  $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$  theory we end up in the SPP theory. We have chosen to describe the general method in order to make explicit how the fine details of the discussion in section 4.5 work. Other more involved examples which better illustrate the main idea can be found in [81].

<sup>19</sup>Note that there is no invariant way of defining A and B in this example, and any other choice of a couple of zig-zag paths with different (p,q) charges will lead to the same IR theory.



**Figure 13.** The A and B zig-zag paths for the  $C^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow SPP$  orbifold.

## References

- [1] S. Kachru and E. Silverstein, *4D conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [[hep-th/9802183](#)] [[SPIRES](#)].
- [2] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [[hep-th/9807080](#)] [[SPIRES](#)].
- [3] D.R. Morrison and M.R. Plesser, *Non-spherical horizons. I*, *Adv. Theor. Math. Phys.* **3** (1999) 1 [[hep-th/9810201](#)] [[SPIRES](#)].
- [4] M. Bertolini, F. Bigazzi and A.L. Cotrone, *New checks and subtleties for AdS/CFT and a-maximization*, *JHEP* **12** (2004) 024 [[hep-th/0411249](#)] [[SPIRES](#)].
- [5] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, *An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, *JHEP* **06** (2005) 064 [[hep-th/0411264](#)] [[SPIRES](#)].
- [6] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, *Commun. Math. Phys.* **262** (2006) 51 [[hep-th/0411238](#)] [[SPIRES](#)].
- [7] D. Martelli, J. Sparks and S.-T. Yau, *The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds*, *Commun. Math. Phys.* **268** (2006) 39 [[hep-th/0503183](#)] [[SPIRES](#)].

- [8] S. Benvenuti and M. Kruczenski, *From Sasaki-Einstein spaces to quivers via BPS geodesics:  $L^{p,q|r}$* , *JHEP* **04** (2006) 033 [[hep-th/0505206](#)] [[SPIRES](#)].
- [9] A. Butti, D. Forcella and A. Zaffaroni, *The dual superconformal theory for  $L(p, q, r)$  manifolds*, *JHEP* **09** (2005) 018 [[hep-th/0505220](#)] [[SPIRES](#)].
- [10] S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh and B. Wecht, *Gauge theories from toric geometry and brane tilings*, *JHEP* **01** (2006) 128 [[hep-th/0505211](#)] [[SPIRES](#)].
- [11] A. Butti and A. Zaffaroni, *R-charges from toric diagrams and the equivalence of  $a$ -maximization and  $Z$ -minimization*, *JHEP* **11** (2005) 019 [[hep-th/0506232](#)] [[SPIRES](#)].
- [12] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and  $\chi$ SB-resolution of naked singularities*, *JHEP* **08** (2000) 052 [[hep-th/0007191](#)] [[SPIRES](#)].
- [13] S. Franco, Y.-H. He, C. Herzog and J. Walcher, *Chaotic duality in string theory*, *Phys. Rev. D* **70** (2004) 046006 [[hep-th/0402120](#)] [[SPIRES](#)].
- [14] S. Franco, A. Hanany, F. Saad and A.M. Uranga, *Fractional branes and dynamical supersymmetry breaking*, *JHEP* **01** (2006) 011 [[hep-th/0505040](#)] [[SPIRES](#)].
- [15] D. Berenstein, C.P. Herzog, P. Ouyang and S. Pinansky, *Supersymmetry breaking from a Calabi-Yau singularity*, *JHEP* **09** (2005) 084 [[hep-th/0505029](#)] [[SPIRES](#)].
- [16] M. Bertolini, F. Bigazzi and A.L. Cotrone, *Supersymmetry breaking at the end of a cascade of Seiberg dualities*, *Phys. Rev. D* **72** (2005) 061902 [[hep-th/0505055](#)] [[SPIRES](#)].
- [17] A. Brini and D. Forcella, *Comments on the non-conformal gauge theories dual to  $Y(p, q)$  manifolds*, *JHEP* **06** (2006) 050 [[hep-th/0603245](#)] [[SPIRES](#)].
- [18] A. Hanany and K.D. Kennaway, *Dimer models and toric diagrams*, [hep-th/0503149](#) [[SPIRES](#)].
- [19] S. Franco, A. Hanany, K.D. Kennaway, D. Vegh and B. Wecht, *Brane Dimers and quiver gauge theories*, *JHEP* **01** (2006) 096 [[hep-th/0504110](#)] [[SPIRES](#)].
- [20] A. Hanany and D. Vegh, *Quivers, tilings, branes and rhombi*, *JHEP* **10** (2007) 029 [[hep-th/0511063](#)] [[SPIRES](#)].
- [21] B. Feng, Y.-H. He, K.D. Kennaway and C. Vafa, *Dimer models from mirror symmetry and quivering amoebae*, *Adv. Theor. Math. Phys.* **12** (2008) 3 [[hep-th/0511287](#)] [[SPIRES](#)].
- [22] K.D. Kennaway, *Brane tilings*, *Int. J. Mod. Phys. A* **22** (2007) 2977 [[arXiv:0706.1660](#)] [[SPIRES](#)].
- [23] M. Yamazaki, *Brane tilings and their applications*, *Fortsch. Phys.* **56** (2008) 555 [[arXiv:0803.4474](#)] [[SPIRES](#)].
- [24] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [[hep-th/0510251](#)] [[SPIRES](#)].
- [25] I. Biswas, D. Gaiotto, S. Lahiri and S. Minwalla, *Supersymmetric states of  $N = 4$  Yang-Mills from giant gravitons*, *JHEP* **12** (2007) 006 [[hep-th/0606087](#)] [[SPIRES](#)].
- [26] D. Martelli, J. Sparks and S.-T. Yau, *Sasaki-Einstein manifolds and volume minimisation*, *Commun. Math. Phys.* **280** (2008) 611 [[hep-th/0603021](#)] [[SPIRES](#)].
- [27] S. Benvenuti, B. Feng, A. Hanany and Y.-H. He, *Counting BPS operators in gauge theories: quivers, syzygies and plethystics*, *JHEP* **11** (2007) 050 [[hep-th/0608050](#)] [[SPIRES](#)].
- [28] D. Martelli and J. Sparks, *Dual giant gravitons in Sasaki-Einstein backgrounds*,

- Nucl. Phys. B* **759** (2006) 292 [[hep-th/0608060](#)] [[SPIRES](#)].
- [29] C.P. Herzog and J. Walcher, *Dibaryons from exceptional collections*, *JHEP* **09** (2003) 060 [[hep-th/0306298](#)] [[SPIRES](#)].
- [30] A. Butti, D. Forcella and A. Zaffaroni, *Counting BPS baryonic operators in CFTs with Sasaki-Einstein duals*, *JHEP* **06** (2007) 069 [[hep-th/0611229](#)] [[SPIRES](#)].
- [31] A. Hanany and C. Romelsberger, *Counting BPS operators in the chiral ring of  $N = 2$  supersymmetric gauge theories or  $N = 2$  brane surgery*, *Adv. Theor. Math. Phys.* **11** (2007) 1091 [[hep-th/0611346](#)] [[SPIRES](#)].
- [32] B. Feng, A. Hanany and Y.-H. He, *Counting gauge invariants: the plethystic program*, *JHEP* **03** (2007) 090 [[hep-th/0701063](#)] [[SPIRES](#)].
- [33] D. Forcella, A. Hanany and A. Zaffaroni, *Baryonic generating functions*, *JHEP* **12** (2007) 022 [[hep-th/0701236](#)] [[SPIRES](#)].
- [34] A. Butti, D. Forcella, A. Hanany, D. Vegh and A. Zaffaroni, *Counting chiral operators in quiver gauge theories*, *JHEP* **11** (2007) 092 [[arXiv:0705.2771](#)] [[SPIRES](#)].
- [35] D. Forcella, *BPS partition functions for quiver gauge theories: counting fermionic operators*, [arXiv:0705.2989](#) [[SPIRES](#)].
- [36] D. Forcella, A. Hanany, Y.-H. He and A. Zaffaroni, *The master space of  $N = 1$  gauge theories*, *JHEP* **08** (2008) 012 [[arXiv:0801.1585](#)] [[SPIRES](#)].
- [37] D. Forcella, A. Hanany, Y.-H. He and A. Zaffaroni, *Mastering the master space*, *Lett. Math. Phys.* **85** (2008) 163 [[arXiv:0801.3477](#)] [[SPIRES](#)].
- [38] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[SPIRES](#)].
- [39] C.E. Beasley, *BPS branes from baryons*, *JHEP* **11** (2002) 015 [[hep-th/0207125](#)] [[SPIRES](#)].
- [40] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *JHEP* **06** (2000) 008 [[hep-th/0003075](#)] [[SPIRES](#)].
- [41] S. Gukov, M. Rangamani and E. Witten, *Dibaryons, strings and branes in AdS orbifold models*, *JHEP* **12** (1998) 025 [[hep-th/9811048](#)] [[SPIRES](#)].
- [42] S.S. Gubser and I.R. Klebanov, *Baryons and domain walls in an  $N = 1$  superconformal gauge theory*, *Phys. Rev. D* **58** (1998) 125025 [[hep-th/9808075](#)] [[SPIRES](#)].
- [43] D. Berenstein, C.P. Herzog and I.R. Klebanov, *Baryon spectra and AdS/CFT correspondence*, *JHEP* **06** (2002) 047 [[hep-th/0202150](#)] [[SPIRES](#)].
- [44] K.A. Intriligator and B. Wecht, *Baryon charges in 4D superconformal field theories and their AdS duals*, *Commun. Math. Phys.* **245** (2004) 407 [[hep-th/0305046](#)] [[SPIRES](#)].
- [45] A. Mikhailov, *Giant gravitons from holomorphic surfaces*, *JHEP* **11** (2000) 027 [[hep-th/0010206](#)] [[SPIRES](#)].
- [46] I.R. Klebanov and A. Murugan, *Gauge/gravity duality and warped resolved conifold*, *JHEP* **03** (2007) 042 [[hep-th/0701064](#)] [[SPIRES](#)].
- [47] D. Martelli and J. Sparks, *Symmetry-breaking vacua and baryon condensates in AdS/CFT*, [arXiv:0804.3999](#) [[SPIRES](#)].
- [48] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys. B* **556** (1999) 89 [[hep-th/9905104](#)] [[SPIRES](#)].



- [49] G.S. Adkins, C.R. Nappi and E. Witten, *Static properties of nucleons in the Skyrme model*, *Nucl. Phys. B* **228** (1983) 552 [SPIRES].
- [50] H. Hata, T. Sakai, S. Sugimoto and S. Yamato, *Baryons from instantons in holographic QCD*, [hep-th/0701280](#) [SPIRES].
- [51] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys. B* **456** (1995) 130 [[hep-th/9507158](#)] [SPIRES].
- [52] E. Witten, *Non-perturbative superpotentials in string theory*, *Nucl. Phys. B* **474** (1996) 343 [[hep-th/9604030](#)] [SPIRES].
- [53] J.A. Harvey and G.W. Moore, *Superpotentials and membrane instantons*, [hep-th/9907026](#) [SPIRES].
- [54] E. Witten, *World-sheet corrections via D-instantons*, *JHEP* **02** (2000) 030 [[hep-th/9907041](#)] [SPIRES].
- [55] R. Blumenhagen, M. Cvetič and T. Weigand, *Spacetime instanton corrections in 4D string vacua — the seesaw mechanism for D-brane models*, *Nucl. Phys. B* **771** (2007) 113 [[hep-th/0609191](#)] [SPIRES].
- [56] L.E. Ibáñez and A.M. Uranga, *Neutrino Majorana masses from string theory instanton effects*, *JHEP* **03** (2007) 052 [[hep-th/0609213](#)] [SPIRES].
- [57] B. Florea, S. Kachru, J. McGreevy and N. Saulina, *Stringy instantons and quiver gauge theories*, *JHEP* **05** (2007) 024 [[hep-th/0610003](#)] [SPIRES].
- [58] O. Aharony and S. Kachru, *Stringy instantons and cascading quivers*, *JHEP* **09** (2007) 060 [[arXiv:0707.3126](#)] [SPIRES].
- [59] R. Argurio, G. Ferretti and C. Petersson, *Instantons and toric quiver gauge theories*, *JHEP* **07** (2008) 123 [[arXiv:0803.2041](#)] [SPIRES].
- [60] D. Krefl, *A gauge theory analog of some 'stringy' D-instantons*, *Phys. Rev. D* **78** (2008) 066004 [[arXiv:0803.2829](#)] [SPIRES].
- [61] L.E. Ibáñez and A.M. Uranga, *Instanton induced open string superpotentials and branes at singularities*, *JHEP* **02** (2008) 103 [[arXiv:0711.1316](#)] [SPIRES].
- [62] C. Beasley and E. Witten, *New instanton effects in supersymmetric QCD*, *JHEP* **01** (2005) 056 [[hep-th/0409149](#)] [SPIRES].
- [63] C. Beasley and E. Witten, *New instanton effects in string theory*, *JHEP* **02** (2006) 060 [[hep-th/0512039](#)] [SPIRES].
- [64] I. Garcia-Etxebarria, F. Marchesano and A.M. Uranga, *Non-perturbative F-terms across lines of BPS stability*, *JHEP* **07** (2008) 028 [[arXiv:0805.0713](#)] [SPIRES].
- [65] R. Blumenhagen, M. Cvetič, R. Richter and T. Weigand, *Lifting D-instanton zero modes by recombination and background fluxes*, *JHEP* **10** (2007) 098 [[arXiv:0708.0403](#)] [SPIRES].
- [66] I. Garcia-Etxebarria and A.M. Uranga, *Non-perturbative superpotentials across lines of marginal stability*, *JHEP* **01** (2008) 033 [[arXiv:0711.1430](#)] [SPIRES].
- [67] R. Blumenhagen and M. Schmidt-Sommerfeld, *Power towers of string instantons for  $N = 1$  vacua*, *JHEP* **07** (2008) 027 [[arXiv:0803.1562](#)] [SPIRES].
- [68] M. Cvetič, R. Richter and T. Weigand, *(Non-)BPS bound states and D-brane instantons*, *JHEP* **07** (2008) 012 [[arXiv:0803.2513](#)] [SPIRES].

- [69] S. Franco and A.M. Uranga, *Dynamical SUSY breaking at meta-stable minima from D-branes at obstructed geometries*, *JHEP* **06** (2006) 031 [[hep-th/0604136](#)] [[SPIRES](#)].
- [70] O.J. Ganor, *A note on zeroes of superpotentials in F-theory*, *Nucl. Phys. B* **499** (1997) 55 [[hep-th/9612077](#)] [[SPIRES](#)].
- [71] D. Baumann, A. Dymarsky, I. R. Klebanov, J.M. Maldacena, L.P. McAllister and A. Murugan, *On D3-brane potentials in compactifications with fluxes and wrapped D-branes*, *JHEP* **11** (2006) 031 [[hep-th/0607050](#)] [[SPIRES](#)].
- [72] M. Berg, M. Haack and B. Körs, *Loop corrections to volume moduli and inflation in string theory*, *Phys. Rev. D* **71** (2005) 026005 [[hep-th/0404087](#)] [[SPIRES](#)].
- [73] Y. Imamura, K. Kimura and M. Yamazaki, *Anomalies and O-plane charges in orientifolded brane tilings*, *JHEP* **03** (2008) 058 [[arXiv:0801.3528](#)] [[SPIRES](#)].
- [74] O. Aharony and A. Hanany, *Branes, superpotentials and superconformal fixed points*, *Nucl. Phys. B* **504** (1997) 239 [[hep-th/9704170](#)] [[SPIRES](#)].
- [75] O. Aharony, A. Hanany and B. Kol, *Webs of  $(p, q)$  5-branes, five dimensional field theories and grid diagrams*, *JHEP* **01** (1998) 002 [[hep-th/9710116](#)] [[SPIRES](#)].
- [76] N.C. Leung and C. Vafa, *Branes and toric geometry*, *Adv. Theor. Math. Phys.* **2** (1998) 91 [[hep-th/9711013](#)] [[SPIRES](#)].
- [77] J.H. Brodie, *Fractional branes, confinement and dynamically generated superpotentials*, *Nucl. Phys. B* **532** (1998) 137 [[hep-th/9803140](#)] [[SPIRES](#)].
- [78] M. Buican and S. Franco, *SUSY breaking mediation by D-brane instantons*, *JHEP* **12** (2008) 030 [[arXiv:0806.1964](#)] [[SPIRES](#)].
- [79] R. Kenyon, *An introduction to the dimer model*, [math.CO/0310326](#).
- [80] R. Kenyon and J.-M. Schlenker, *Rhombic embeddings of planar graphs with faces of degree 4*, [math.PH/0305057](#).
- [81] I. Garcia-Etxebarria, F. Saad and A.M. Uranga, *Quiver gauge theories at resolved and deformed singularities using dimers*, *JHEP* **06** (2006) 055 [[hep-th/0603108](#)] [[SPIRES](#)].